Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

The harmonic response plot

We will determine whether a linear system (a second order differential equation in this example) is causal by means of the harmonic response plot, from which can be found the number of poles in the lower half complex plane of the harmonic response function using the argument principal. The linear system is causal if there are no poles in the lower half plane.

The argument principle says that if $\tilde{G}(\omega)$ is analytic within a closed curve C in the complex ω-plane, except for a finite number of simple poles, and if $\tilde{G}(\omega)$ has no has no poles or zeros on C, then when C is traversed once in the positive sense

$$\triangle \arg(\tilde{G}(\omega)) = 2\pi(\#\text{poles within } C - \#\text{zeros within } C).$$

To determine whether $\tilde{G}(\omega)$ has poles in the lower half plane, we take C to consist of the real axis traversed from left to right together with a large semicircle in the lower half plane traversed clockwise and calculate geometrically the change in $\arg \tilde{G}$.

We take

$$\tilde{G}(\omega) = \frac{1}{-\omega^2 + 2ib\omega + \omega_0^2}$$

which is the response of the differential equation

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = f(t)$$

to the input $f(t) = e^{i\omega t}$. In this equation, b and ω_0 are real.

We write $\tilde{G}(\omega) = A(\omega)e^{i\phi(\omega)}$ where

$$\cos\phi = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0 - \omega^2)^2 + 4b^2}} \qquad \text{and} \qquad \sin\phi = \frac{-2b\omega}{\sqrt{(\omega_0 - \omega^2)^2 + 4b^2}}.$$

For the large semi-circle $\omega = Re^{i\theta}$, $(0 \geq \theta \geq -\pi)$ we have $\tilde{G}(\omega) \approx -R^{-2}e^{-2i\theta}$, so the semi-circle in the ω-plane corresponds to a small anti-clockwise circle on the harmonic response plot with $\phi = 0$ at the start and $\phi = 2\pi$ at the end.

On the real axis, starting at $\omega = -R$ and $\phi = 2\pi$, we see that $\sin\phi$ starts small and negative if $b > 0$ but small and positive if $b < 0$. As the axis is traversed, $\sin\phi$ passes through zero again once ending up small and positive if $b > 0$ but small and negative if $b < 0$. Thus ϕ decreases or increases by (very nearly) 2π according to the sign of b.

For $b > 0$, the winding number is 0 so there are no poles in the lower half plane and the system is causal. For $b < 0$, the winding number is 2 so there are two poles in the lower half plane and the system is acausal.