Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

The Hankel representation of $\Gamma(z)$.

Unlike many handouts, this one is not a worked example; it is the proof of a very important result.

The Hankel representation (defined below) for $\Gamma(z)$ is valid for all z except $z = 0, -1, -2, \ldots$, and so provides the analytic continuation of $I(z)$, the Euler integral representation of $\Gamma(z)$. The derivation of the Hankel representation is a prototype for integral representations of many other functions.

The *Hankel representation* of $\Gamma(z)$ is

$$\Gamma(z) = \frac{1}{2i\sin\pi z}\int_{-\infty}^{(0^+)} e^t\, t^{z-1}\,\mathrm{d}t \qquad (z \neq 0, -1, -2, \ldots) \tag{$*$}$$

where $|\arg t| \leq \pi$ and the path of integration is the *Hankel contour* as shown below. The notation (0^+) means that the path goes round the origin in the positive sense.

The integral in equation $(*)$ defines an analytic function: its integrand is analytic in z; the convergence at the ends of the path is exponential for all z; and the Hankel contour can be kept well clear of the singularity at the origin of the t-plane.

To demonstrate that the Hankel representation provides the required analytic continuation, we need to show that it is equal to $I(z)$ for $Re\ z > 0$. This we can do by collapsing the contour onto the branch cut, the integral converging at $t = 0$ (or, equivalently, the contribution from the small circle round the origin vanishing) because of the condition on $Re\ z$.

Let

$$J(z) = \int_{-\infty}^{(0^+)} e^t\, t^{z-1}\,\mathrm{d}t.$$

When $Re\ z > 0$ we can write

$$\begin{aligned} J(z) &= \int_{\gamma_1} e^t\, t^{z-1}\,\mathrm{d}t + \int_{\gamma_2} e^t\, t^{z-1}\,\mathrm{d}t + \int_{\gamma_3} e^t\, t^{z-1}\,\mathrm{d}t \\ &\equiv J_1(z) + J_2(z) + J_3(z) \end{aligned}$$

where the paths γ_1, γ_2 and γ_3 are given by

γ_1: $t = xe^{i\pi}$; $\quad \epsilon \leq x < \infty$ (above the cut)

γ_2: $t = xe^{-i\pi}$; $\quad \infty > x \geq \epsilon$ (below the cut)

γ_3: $t = \epsilon e^{i\theta}$; $\quad -\pi < \theta < \pi$ (round the small circle)

As $\epsilon \to 0$, we have

$$\begin{aligned} J_1(z) &\to (e^{i\pi})^z \int_0^\infty e^{-x} x^{z-1}\,\mathrm{d}x \\ J_2(z) &\to (e^{-i\pi})^z \int_\infty^0 e^{-x} x^{z-1}\,\mathrm{d}x. \\ J_3(z) &= \int_{-\pi}^{\pi} e^{\epsilon e^{i\theta}} (\epsilon e^{i\theta})^{z-1}\,\mathrm{d}(\epsilon e^{i\theta}) \to 0. \end{aligned}$$

The last integral, being proportional to ϵ^z, tends to zero since $Re\ z > 0$. Adding the integrals establishes the required result:

$$J(z) = 2i \sin \pi z I(z) \qquad (Re\ z > 0).$$

From the Hankel representation it seems at first sight that $\Gamma(z)$ has simple poles whenever z is an integer, due to the factor $\sin \pi z$ in the denominator. There are certainly no other singularities since the integral $J(z)$ is entire. But $\Gamma(z)$ is already known, from the integral $I(z)$, to be analytic for $Re\ z > 0$, so the simple zeros of $\sin \pi z$ must be cancelled by zeros of the integral when z is a positive integer.

This can be seen as follows. When z is a positive integer, the integrand of $J(z)$ has no singularity at the origin. There are no singularities within the Hankel contour: no branch point and therefore no branch cut. Thus $J(z)$ vanishes. The zero of $J(z)$ cancels the zero of $\sin \pi z$ giving a finite result.[1]

We can calculate the residues at the poles easily, because when z is an integer, the branch cut in the t-plane introduced on account of t^{z-1} is no longer needed, and the Hankel contour can be reduced to a circle round the origin. Setting $z = -m \leq 0$, we have

$$J(-m) = \int_{|t|=1} e^t t^{-m-1}\,\mathrm{d}t = \frac{2\pi i}{m!}.$$

The residue was calculated by expanding the exponential. The residue of $\Gamma(z)$ at $z = -m$ is therefore

$$\lim_{z \to -m} \left(\frac{z+m}{2i \sin \pi z} J(z) \right) = \frac{2\pi i}{m!} \lim_{z \to -m} \left(\frac{z+m}{2i \sin \pi z} \right) = \frac{(-1)^m}{m!},$$

where l'Hôpital's rule was used to find the limit.

[1] In general, this argument would be rather suspect. What interpretation are we to place on zero divided by zero? In the case of analytic functions, we can expand top and bottom in their series representations and since we know that $\sin \pi z$ has a simple zero at $z = m$, it is clear that in the limit as $z \to m$ the ratio either tends to a constant or zero.