Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Example 1.3: Functions defined by integrals

(i) We investigate the function $F(z)$ defined by the gaussian-type integral

$$F(z) = \int_{-\infty}^{\infty} e^{-zt^2} dt.$$

We pretend that we can't evaluate the integral[1].

First we decide for which values of z $F(z)$ is defined. Since the integrand is a nice smooth function of both t and z, the only problem lies in the convergence[2]: the integral between finite limits would be defined for all z. Clearly, the integral converges if $\Re z > 0$ and does not converge if $\Re z < 0$. The only uncertainty occurs when z is imaginary. In this case, it seems that it might not converge (since the integral of $\exp(it)$ certainly does not). It does not converge absolutely, since $|e^{-zt^2}| = 1$.

In fact, the integral does converge when z is pure imaginary, as can be seen by integrating by parts. We consider the integral from 1 to R, since nothing goes wrong with the convergence of the integral from 0 to 1 (but a lower limit of 0 creates problems in the integration by parts):

$$\int_1^R e^{-zt^2} dt = \int_1^R \frac{te^{-zt^2}}{t} dt = -\frac{e^{-zt^2}}{2t}\bigg|_1^R - \int_1^R \frac{e^{-zt^2}}{2t^2} dt.$$

Both parts of this last expression are well-behaved in the limit $R \to \infty$, so the integral does indeed converge.

Now we tackle the question of analyticity. Our first instinct would be to say simply that since $F(z)$ is defined for $\Re z \geq 0$, it is analytic for $\Re z > 0$[3]. This instinct is correct.

We might next think about the conditions of the theorem. Is the integrand smooth (jointly continuous in z and t, analytic in z)? Yes — obviously. Is it the case that the tails don't matter? Yes — they are exponentially small when $\Re z > 0$. Therefore $F(z)$ is holomorphic for $\Re z > 0$.

If we had all day to think about this, we might use the Weierstrass test to determine uniform convergence. This involves finding a finding a function that dominates the integrand but which does not depend on the parameter z. For $Re\ z > \epsilon$, we have

$$|\exp(-zt^2)| < \exp(-\epsilon t^2)$$

and this last function can be integrated. Thus $F(z)$ is analytic for $Re\ z > \epsilon$, for any positive ϵ, which is the same as saying that it is analytic for $\Re z > 0$.

(ii) Let

$$F(z) = \int_0^\infty \frac{u^{z-1}}{u+1} du$$

We first investigate for which range of values of z for which $F(z)$ is defined.

[1] We can: a change of variable and a rotation of the resulting path back onto the real axis relates it to a standard gaussian integral; the result is $(\pi/z)^{1/2}$.

[2] Recall that an infinite integral is defined as a limit: $\displaystyle\int_0^\infty f(t)dt = \lim_{R\to\infty} \int_0^R f(t)dt$. The integral converges when this limit exists (just like an infinite sum).

[3] Remember that analyticity requires open sets.

Clearly, the only problems that might arise are associated with $u = 0$ and $u = \infty$. The integrand is well behaved elsewhere: it is jointly continuous in u and z, since $u = -1$ is not relevent; and it is analytic in z, as can be seen by writing the denominator as $e^{(z-1)\log u}$.

At $u = 0$, there is potentially a problem when $\Re z \leq 1$, because then the integrand is not continuous (it is undefined), and the integral must be defined as a limit:

$$\int_0 \frac{u^{z-1}}{u+1} du = \lim_{\epsilon \to 0} \int_\epsilon \frac{u^{z-1}}{u+1} du$$

However, near $u = 0$, the integrand is approximately u^{z-1}, which can be integrated to give $z^{-1}u^z$. In the limit $\epsilon \to 0$, this gives a (finite) limit provided $\Re z > 0$. This endpoint therefore places the restriction $\Re z > 0$ on $F(z)$.

Now for the infinite end. Again, we must define the integral as a limit ($R \to \infty$, where R is the upper end point). There is no difficulty if $\Re z < 0$ since we can easily see that integral converges. We can again do better by means of an approximation. We take

$$\frac{u^{z-1}}{u+1} \approx u^{z-2}.$$

Integrating gives $(z-1)^{-1}u^{z-1}$, which will converge provided $\Re z < 1$.

Combining these two results, we see that $F(z)$ is defined for $0 < \Re z < 1$.

Now we have to decide the range of z for which $F(z)$ is analytic. We would be astonished if this were not $0 < \Re z < 1$. If we really want to check it, we could split the integral:

$$F(z) = \int_0^1 \frac{u^{z-1}}{u+1} du + \int_1^\infty \frac{u^{z-1}}{u+1} du.$$

Now integrate the first integral once by parts, so that u^z appears in the new integrand. The integrand is now certainly smooth for $\Re z > 0$ and $0 \leq u \leq 1$ and analytic in z for $z \neq 0$, so the integral is analytic. For the second integral, we only have to determine whether it is uniformly convergent. Since the integrand $\approx u^{z-2}$ for large u, contributions to the integral from $u \gg 1$ will certainly by small for $\Re z < 1$.

Therefore, the $F(z)$ is analytic for $0 < \Re z < 1$.

We can verify this by integration. (It is a result that is required for proving the result $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z$.)

Let $J = \displaystyle\int_\gamma \frac{t^{z-1}}{t+1} dt = J_1 + J_2 + J_3 + J_4$, where J_i is the integral along γ_i, and

$\gamma_1 :$	$t = u$	$\epsilon \leq u \leq R$
$\gamma_2 :$	$t = ue^{2\pi i}$	$R \geq u \geq \epsilon$
$\gamma_3 :$	$t = \epsilon e^{i\theta}$	$2\pi \geq \theta \geq 0$
$\gamma_4 :$	$t = Re^{i\theta}$	$0 \leq \theta \leq 2\pi$

The integrals round the small and large circles tend to zero as $\epsilon \to 0$ and $R \to \infty$; the condition for this is precisely $0 < \Re z < 1$. There is a simple pole at $t = e^{i\pi}$, with residue $(e^{i\pi})^{z-1}$. Thus

$$2\pi i (e^{i\pi})^{z-1} = I - I(e^{2\pi i})^{z-1} \Longrightarrow I = \pi \operatorname{cosec} \pi z$$

which is certainly analytic on $0 < \Re z < 1$.