Example: Functions defined by integrals

We will investigate the range of values of z for which a function defined by an integral exists and for which it is analytic. No serious analysis is necessary: the most important thing is to understand when the integral converges and when it doesn't.

Let

$$F(z) = \int_0^\infty \frac{u^{z-1}}{u+1} du$$

(i) Existence

We first investigate for which range of values of z for which $F(z)$ exists. Clearly, the only problems that might arise are associated with $u = 0$ and $u = \infty$. The integrand is well-behaved elsewhere: $u = -1$ is not in the range of integration and there are no values of z which cause problems (as can be seen by writing the numerator as $e^{(z-1)\log u}$).

To investigate possible problems at $u = 0$, we approximate $u + 1$ by 1 and consider the integral

$$\int_0 \frac{u^{z-1}}{1} du = \frac{u^z}{z}\bigg|_0$$

(ignoring the upper limit; we deal with that next). Now

$$|u^z| = \left|e^{z \log u}\right| = e^{x \log u}$$

which can be evaluated at $u = 0$ provided $x > 0$, i.e. provided $\Re z > 0$.

Now for the infinite end. This time, we approximate $u + 1$ by u and consider the integral

$$\int^\infty \frac{u^{z-1}}{u} du = \frac{u^{z-1}}{z+1}\bigg|^\infty$$

(ignoring the lower limit, which we have dealt with). Now

$$\left|u^{z-1}\right| = \left|e^{(z-1)\log u}\right| = e^{(x-1)\log u}$$

which can be evaluated at $u = \infty$ provided $x < 1$, i.e. provided $\Re z < 1$.

Clearly, if $\Re z < 0$ or $\Re z > 1$ the integral does not converge at one of the endpoints. But what about the cases $\Re z = 0$ or $\Re z = 1$? If z or $z - 1$ is imaginary, the integrand oscillates rapidly as the endpoints are approached and these rapid oscillations might provide cancellation which allow the integral to be evaluated. This is what happened for the Gaussian integral, and we integrated by parts to establish the result. We shall not investigate this possibility further, except to say that the oscillations are not sufficiently fast in this case, and integrating by parts does not help at all.

We therefore conclude that $F(z)$ is defined if and only if $0 < \Re z < 1$.

(ii) Analyticity

Now we have to decide the range of z for which $F(z)$ is analytic. We would be astonished if this were not $0 < \Re z < 1$ (i.e. on any open subset of the range for which the integral exists).

If we really want to check it, we need to check three conditions:

(i) Is the integrand continuous (jointly in u and z)?

(ii) Does the integrand converge uniformly in each compact subset of $0 < \Re z < 1$?

(iii) Is the integrand analytic for each u?

We are not going to prove rigorously that these conditions hold: it is not in the spirit of the course and there is a real danger of missing the wood for the trees.

Instead, note that there is no difficulty about (i) and (iii), since the integrand (writing the numerator as an exponential) is clearly as nice as you would wish for.

What about uniform convergence? Recall that the integral $\int_0^\infty f(z,t)dt$ is uniformly convergent for $z \in U$ if given ϵ, there exists B_0 such that if, $B_0 < B_1 < B_2$ then

$$\left| \int_{B_1}^{B_2} f(z,t)dt \right| < \epsilon,$$

i.e. the tail of the integral doesn't matter.

There are tests we could apply quite easily (e.g. the Weierstrass test). However, it is clear that in the range $0 < a \leq \Re z \leq b < 1$, the rate of convergence is at least as fast as at $z = a$ or $z = b$, and so is under control: the tail does not matter whatever the value of z.

Therefore, the $F(z)$ is analytic for $0 < \Re z < 1$.

Evaluation of the integral

We can evaluate $F(z)$ explicitly by doing the integral. The result of this integration is required for proving the result $\Gamma(z)\Gamma(1-z) = \pi \text{cosec}\, \pi z$ required later on, so it is worth doing.

Let $J = \displaystyle\int_\gamma \frac{t^{z-1}}{t+1} dt = J_1 + J_2 + J_3 + J_4$, where J_i is the integral along γ_i, and

$$\begin{array}{lll} \gamma_1 : & t = u & \epsilon \leq u \leq R \\ \gamma_2 : & t = ue^{2\pi i} & R \geq u \geq \epsilon \\ \gamma_3 : & t = \epsilon e^{i\theta} & 2\pi \geq \theta \geq 0 \\ \gamma_4 : & t = Re^{i\theta} & 0 \leq \theta \leq 2\pi \end{array}$$

The integrals round the small and large circles tend to zero as $\epsilon \to 0$ and $R \to \infty$; the condition for this is precisely $0 < \Re z < 1$. There is a simple pole at $t = e^{i\pi}$, with residue $(e^{i\pi})^{z-1}$. Thus

$$2\pi i (e^{i\pi})^{z-1} = I - I(e^{2\pi i})^{z-1} \Longrightarrow I = \pi \text{cosec}\, \pi z$$

which is certainly analytic on $0 < \Re z < 1$.