Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Solution of differential equations by series

This is largely revision of the material in Part IA Differential Equations, using the language of complex variables. You will not be asked to solve differential equations by series; but it is important to know that this can be done. It is essential to understand the nature of the resulting solutions, and in particular the significance of the the indicial equation and its solutions.

We will investigate the solutions, in the neighbourhood of $z = 0$, of the equation

$$w'' + p(z)w' + q(z)w = 0. \qquad (*)$$

Recall first that $z = 0$ is called an *ordinary point* of the equation if both $p(z)$ and $q(z)$ are analytic at $z = 0$ and that otherwise $z = 0$ is a *singular point* of the equation. Some singular points are more benign than others; in particular, if both $zp(z)$ and $z^2q(z)$ are analytic at $z = 0$, then $z = 0$ is called a *regular singular point* of the equation.

Solutions near an ordinary point

Theorem If $p(z)$ and $q(z)$ are analytic in the disc $|z| < R$, then there exist two linearly independent solutions of $(*)$, $w_1(z)$ and $w_2(z)$, such that:

- $w_1(z)$ and $w_2(z)$ are analytic in $|z| < R$ (and possibly in a larger disc);
- $w_1(0) \neq 0$; $w_2(0) = 0$ and $w_2'(0) \neq 0$ (i.e. the roots of the indicial equation are 0 and 1).

Remarks

1. Note that (for example) z and z^2 cannot both satisfy an equation of the form $(*)$ for which $z = 0$ is an ordinary point.
2. An example of an equation with solutions that are analytic in a disc larger than the disc in which coefficients $p(z)$ and $q(z)$ are analytic is

$$w'' - 2\frac{1}{z-1}w' + 2\frac{1}{(z-1)^2}w = 0$$

   which has solution $A(z-1) + B(z-1)^2$, an entire function despite the singular point of the equation at $z = 1$.
3. We can understand the result in the second bullet point by noting that, for $|z| \ll 1$, equation $(*)$ is approximately $w'' + p_0w' + q_0w = 0$ for which solutions are of the form either $\exp(\alpha z)$ and $\exp(\beta z)$ (in which case linear combinations can be taken to satisfy the required conditions) or $\exp(\alpha z)$ and $z\exp(\alpha z)$ in which case the required conditions are satisfied.

The theorem can be proved by substituting $p(z) = \sum_{n=0}^{\infty} p_n z^n$, $q(z) = \sum_{n=0}^{\infty} q_n z^n$ and $w(z) = \sum_{n=0}^{\infty} a_n z^n$ into the differential equation, equating coefficients of z^n and checking the radius of convergence. Please don't attempt it. (In any case, there are better ways: try googling Picard.)

Solutions near a regular singular point

Suppose $zp(z)$ and $z^2q(z)$ are analytic in the disc $|z| < R$ and let

$$p(z) = \sum_{-1}^{\infty} p_n z^n, \qquad q(z) = \sum_{-2}^{\infty} q_n z^n.$$

The *indicial equation* is

$$\sigma^2 + (p_{-1} - 1)\sigma + q_{-2} = 0.$$

Informally, it can be thought of in connection with the $|z| \ll 1$ approximation to $(*)$, namely

$$w'' + p_{-1}z^{-1}w' + q_{-2}z^{-2}w = 0$$

which has solutions $Az^{\sigma_1} + Bz^{\sigma_2}$, where σ_1 and σ_2 are the roots of the indicial equation above, or $z^{\sigma}(A + B\log z)$ if these roots are equal (note the need for the log in this case).

Theorem Let $z = 0$ be a regular singular point of the equation $(*)$. Then there exist two linearly independent solutions $w_1(z)$ and $w_2(z)$ such that:

- either $w_1(z) = z^{\sigma_1}u_1(z)$ and $w_2(z) = z^{\sigma_2}u_2(z)$ where $u_i(z)$ are analytic for $|z| < R$ (and maybe in a larger disc) and $u_i(0) \neq 0$;
- or $\sigma_1 = \sigma_2 + N$, where N is a non-negative integer, and $w_1(z) = z^{\sigma_1}u_1(z)$ and $w_2(z) = z^{\sigma_2}u_2(z) + w_1(z)\log z$ where $u_i(z)$ are analytic in for $|z| < R$ (and maybe in a larger disc) and $u_i(0) \neq 0$.

To prove this theorem, set $w(z) = z^{\sigma}\sum\limits_{0}^{\infty} a_n z^n$, where the arbitrariness in σ is removed by choosing $a_0 \neq 0$. Substituting into $(*)$ and equating the coefficient of $z^{n+\sigma}$ to zero gives

$$a_nF(n+\sigma) = -\sum_{k=0}^{n-1} a_k[(k+\sigma)p_{n-k-1} + q_{n-k-2}] \quad (n > 0) \qquad (1)$$

$$a_0F(\sigma) = 0 \qquad (2)$$

where

$$F(x) \equiv x(x-1) + p_{-1}x + q_{-2} \equiv (x - \sigma_1)(x - \sigma_2).$$

Since $a_0 \neq 0$, equation (**??**) is exactly the indicial equation. The exponents (i.e. the roots of this quadratic) satisfy

$$\sigma_1 + \sigma_2 = 1 - p_{-1}, \qquad \sigma_1\sigma_2 = q_{-2}.$$

If they do not differ by an integer, then the recurrence relations (**??**) will determine two linearly independent solutions which can be written in the form given in the first bullet point.

If $\sigma_1 = \sigma_2 + N$ $(N = 1, 2, ...)$, the recurrence relations will give one solution corresponding to σ_1 but will usually break down for the smaller root when the coefficient of a_N vanishes (because $F(N + \sigma_2) = F(\sigma_1) = 0$). (It may happen that the remainder of the Nth recurrence relation also vanishes, in which case there is no problem.) Then (and always if $N = 0$) it is necessary to seek a log solution of the form

$$w(z) = z^{\sigma_1}\log z\sum_{0}^{\infty} a_n z^n + z^{\sigma_2}\sum_{0}^{\infty} b_n z^n$$

where the a_n have already been determined. Substituting into the differential equation will give recurrence relations for b_n. (The logs should cancel identically.) This may not be a very pleasant task.

The final step of the proof would be to check radii of convergence. Again, there are better ways.