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Cauchy Principal Value

This hand-out has two examples of integrals involving the Cauchy Principal Value. In the first, the integral only makes sense as a CPV, because the integrand has a simple pole on the path of integration. In the second, the CPV is used as a tool for evaluating quickly a familiar integral. Both examples use the complex plane, but it should be borne in mind that this is merely a device for calculating: there is nothing intrinsically complex about the CPV.

Example (i)

We will evaluate the principal value integral I defined by

$$I = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x}\,\mathrm{d}x$$

where $f(z)$ is analytic in the upper half-plane and $|f(z)| \to 0$ as $|z| \to \infty$.

Consider the integral

$$\begin{aligned} J = \int_C \frac{f(z)}{z}\,\mathrm{d}z \quad &\equiv \quad \int_{\gamma_1} \frac{f(z)}{z}\,\mathrm{d}z + \int_{\gamma_2} \frac{f(z)}{z}\,\mathrm{d}z + \int_{\gamma_3} \frac{f(z)}{z}\,\mathrm{d}z \\ &\equiv \quad J_1 + J_2 + J_3, \end{aligned}$$

where the paths γ_i are as shown below and are given by

γ_1: $z = x$; $\{-R \le x \le -\epsilon\} \cup \{\epsilon \le x \le R\}$

γ_2: $z = \epsilon e^{i\theta}$; $\pi \ge \theta \ge 0$

γ_3: $z = Re^{i\theta}$; $0 \le \theta \le \pi$.

In the limits $\epsilon \to 0$ and $R \to \infty$, we have $J_1 \to I$ (by definition of the principal value), $J_3 \to 0$ and

$$J_2 = \lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{f(\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\,\mathrm{d}\theta = \int_{\pi}^{0} i f(0)\,\mathrm{d}\theta = -i\pi f(0).$$

Also, $J = 0$, since C is a closed contour and the integrand is holomorphic inside and on the contour.

Therefore, $I = i\pi f(0)$.

Remarks

1. If we had chosen γ_2 to be a semi-circle *below* instead of *above* the origin, we would have had $J_2 = +i\pi f(0)$ and $J = 2\pi i f(0)$, so the result would have been the same (of course).

2. The value of I depends only on the value of $f(x)$ for real x (of course); giving the properties of f as a function on the complex plane was just a convenient (though indirect) way of restricting the class of functions for the purposes of this particular example. Note that the restriction does not just affect the way $f(x)$ decays as $x \to \infty$: the form of the answer shows that f cannot be real-valued (unless $f(0) = 0$).

Example (ii)

We will evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} dx.$$

Note that the integrand has only a removable singularity at $x = 0$, and is therefore integrable.

Standard method

First, consider evaluating the integral by standard technique. We would like write the cosine in terms of exponentials, then split the integrand into two parts, one containing e^{+ix} and the other e^{-ix}, so that one contour could be closed in the lower half plane and the other in the upper half plane. However, this would mean having terms like $x^{-2}e^{-ix}$ in the integrand, and the integral would not converge at $x = 0$ on the real axis.

One way round this difficulty is to deform the contour of integration clear of the origin, either above or below, and then carry out the above plan. The integrand is an entire function, so deforming the contour makes no difference to the value of the integral.

Thus $I = \displaystyle\int_\gamma \frac{1}{z^2} dz - \int_\gamma \frac{e^{iz}}{2z^2} - \int_\gamma \frac{e^{-iz}}{2z^2}$.

For the second integral, we close γ in the upper half plane giving zero (since there are no singularities within the closed contour). For the third integral, we close γ in the lower half plane giving $-2\pi i(-i/2)$ (the minus because we are circling the pole or order two in the clockwise sense). For the first integral, we can close either in the upper half or the lower half plane, giving 0 (since in the first case the contour encloses no singularities and in the second case the residue of the enclosed singularity is 0). Therefore, $I = \pi$.

Using the CPV

Note first that $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} dx. = \mathcal{P}\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} dx = \mathrm{Re}\, \mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2} dx$

Consider the integral

$$J = \int_\gamma \frac{1-e^{iz}}{z^2} dz$$

where γ runs along the real axis and is indented over the origin. The integrand has no singularities in the upper half plane so, closing the contour with a large semicircle in the upper half plane, we have $J = 0$.

But splitting γ into a path along the real axis excluding the origin plus an infinitesimal semicircle round the origin gives

$$J = \mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2} dx - i\pi(-i).$$

Thus

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2} dx = \pi$$

and taking real and imaginary parts gives

$$I = \pi, \qquad \mathcal{P}\int_{-\infty}^{\infty} \frac{-\sin x}{x^2} dx = 0.$$