Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Integration using a branch cut

This is an example of the way the complex plane for a many-valued function can be used to evaluate a real integral. It is an old hand-out and this year I will — I expect — go through the example on the board, since it is closely related to the discussion of the different branches of $(1-z^2)^{\frac{1}{2}}$.

We will evaluate the integral

$$I = \int_{-1}^{1} (1-x^2)^{\frac{1}{2}}\, dx.$$

using contour integration.

Let $f(z)$ be the branch of $(1-z^2)^{\frac{1}{2}}$ defined by a branch cut from -1 to 1 with $f(0) = +1$ on the top side of the cut.

Let

$$J = \int_C f(z)\, dz,$$

where the contour C is the circle $|z| = R > 1$, traversed clockwise.

We can evaluate J by integrating the Laurent expansion for f:

$$f(z) = -iz(1-z^{-2})^{\frac{1}{2}} = -iz(1 - \tfrac{1}{2}z^{-2} + \cdots),$$

so

$$J = \int_{C'} (-iz)(1 - \tfrac{1}{2}z^{-2} + \cdots)\, dz = \int_0^{-2\pi} (-iRe^{i\theta})(1 - \tfrac{1}{2}R^{-2}e^{-2i\theta} + \cdots) iRe^{i\theta}\, d\theta = \pi$$

since all terms in the Laurent expansion except the second contribute are periodic. This is of course just the residue term, even though the singularities within C are not poles (they are branch points, and discontinuities at each point on the cut).

This result could also be obtained by setting $t = 1/z$ and using the residue theorem on the pole at the origin:

$$J = \int_C (1-z^2)^{\frac{1}{2}} dz = -\int_{C'} (1-t^{-2})^{\frac{1}{2}} t^{-2} dt = -\int_{C'} -it^{-3}(1-t^2)^{\frac{1}{2}} dt = \pi.$$

where C' is the circle $|t| = 1/R$ traversed anticlockwise. This method is equivalent to calculating the residue at the pole *outside* the contour (at infinity).

We can also obtain a value for J by collapsing the contour onto the branch cut:

$$J = \int_{-1}^{1} (1-x^2)^{\frac{1}{2}} dx + \int_1^{-1} -(1-x^2)^{\frac{1}{2}} dx = 2I.$$

Comparing this with the previous value shows that $I = \pi/2$.

Remark Of course, it cannot matter what branch of $f(z)$ is used, but some choices might need a cunning choice of contour. The branch defined by a cut along the real axis *excluding* the segment $-1 < x < 1$ could be used in conjunction, for example, with a large semi-circular contour, for which the contributions to the integral from the two sections of the cut would cancel.