Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

The arcsin function defined as an integral

We will define a (single-valued) function Arcsin by means of an integral, from which can be constructed the (mulitvalued) arcsin function by analytic continuation. The periodic property of the inverse function (sin) can be inferred. This is analogous to the construction of inverse elliptic functions from integrals.

Let

$$\operatorname{Arcsin} z = \int_0^z \frac{dt}{(1-t^2)^{\frac{1}{2}}}$$

where

- the branch of $(1-t^2)^{\frac{1}{2}}$ is defined by a cut (of length 2) along the real axis joining $t=-1$ and $t=1$ with $(1-t^2)^{\frac{1}{2}}=+1$ at the origin above the cut (i.e at $t=0^+$);
- the path of integration is a straight line from $t=0^+$ to $t=z$ if $0 \le \arg z \le \pi$ and a path anticlockwise round the branch cut from $t=0^+$ to $t=0^-$, followed by a straight line from $t=0^-$ to $t=z$ if $\pi < \arg z < 2\pi$.

Before we consider analytically continuing $\operatorname{Arcsin} z$, we must determine the domain on which it is analytic. Note first that we can differentiate the integral trivially, to obtain $(1-z^2)^{-\frac{1}{2}}$, this branch being defined as in the first bullet point above for $(1-t^2)^{\frac{1}{2}}$.

This means that the only places where Arcsin is not analytic occur where $(1-z^2)^{-\frac{1}{2}}$ is badly behaved (i.e. on the branch cut) and at the boundary of the region for which Arcsin is defined[1] (i.e. on the positive real axis $\arg z = 0$). Thus Arcsin is not analytic on the real axis for $-1 \le x < \infty$; and in fact, Arcsin is discontinuous across this section of the axis (as can easily be verified), which is therefore a branch cut.

Now consider a new function $\operatorname{Arcsin}_1(z)$ defined by the same integral as Arcsin, except that now the path of integration is a path anticlockwise round the branch cut from $t=0^+$ to $t=0^-$, followed by a straight line from $t=0^-$ to $t=z$ if $\pi < \arg z < 2\pi$ and a path anticlockwise all the way round the branch cut from $t=0^+$ back to $t=0^+$, followed by a straight line from $t=0^+$ to $t=z$, if $2\pi < \arg z < 3\pi$:

[1] Compare with functions of a real variable: such a function defined on a closed interval cannot be differentiable at the endpoints.

Clearly $\mathrm{Arcsin}_1(z)$ is obtained by analytic continuation: it defines a function analytic on $\pi < \arg z < 3\pi$ and agrees with $\mathrm{Arcsin}(z)$ when the domains of definition of the two functions overlap ($\pi < \arg z < 2\pi$).

This process can be repeated to obtain a multivalued function, $\arcsin(z)$, which has a discontinuity across the real axis for $-1 \le x \le 1$. The dependence of $\arcsin z$ on $\arg z$ is similar to that of $\log z$ as can easily be seen: for example, if $0 \le \arg z \le 2\pi$,

$$\arcsin\left(e^{2\pi i} z\right) = \mathrm{Arcsin}\, z + \int_{\text{round cut}} \frac{dt}{(1-t^2)^{\frac{1}{2}}} = \mathrm{Arcsin}\, z - 2\pi.$$

In general, the possible values of $\arcsin z$ are $\mathrm{Arcsin}\, z + 2\pi n$, where n is any integer.

Now let us consider the function that is inverse to arcsin, without worrying too much about existence problems; call it sin. We have

$$\sin\left(\arcsin\left(e^{2\pi i} z\right)\right) = \sin(\mathrm{Arcsin\, z} - 2\pi).$$

But $\sin\left(\arcsin\left(e^{2\pi i} z\right)\right) = e^{2\pi i} z = z$, so setting $w = \mathrm{Arcsin\, z}$ gives

$$\sin w = \sin(w - 2\pi).$$

We see that $\sin w$ defined in this way is periodic with period 2π.

Remark It is unfortunate that, according to this definition, $\arcsin z$ is not analytic on the real axis for $-1 \le x \le 1$ which is just where we might have wanted it to be well behaved. An alternative plan would have been to define

$$\arcsin z = \int_0^z \frac{dt}{(1-t^2)^{\frac{1}{2}}}$$

where the path of integration is any path at all, not crossing $t = \pm 1$ but winding about these points any number of times, and where the value of the integrand at any point on this path is obtained by analytic continuation along the path from an initial value of $+1$ at $z = 0$. This would yield a multivalued (path-dependent) function with branch points at $z = \pm 1$ but no discontinuities.

Remark The inverse elliptic functions can be defined in an analogous way as integrals with the square root of quartic function in the denominator. Analytic continuation yields a multivalues function with values differing by constants of the form $Km + iK'n$, for integers m and n. This shows that the related elliptic functions are doubly periodic.