Mathematical Tripos Part II Michaelmas term 2007
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Analytic continuation by contour deformation

Let

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} dt \qquad Im\ z > 0.$$

The purpose of this example is to show how $F(z)$ (whose domain, by definition, is the upper half plane only) can be analytically continued to the lower half plane. We do this by defining a new function, namely an integral with the same integrand but a different path of integration. The idea is that this new integral will agree with $F(z)$ in regions of the complex plane where they are both defined but will be defined in some parts of the complex plane where $F(z)$ is not defined, thereby providing an analytic continuation.

The first question to answer is why the domain of $F(z)$ is restricted to Im $z > 0$. Clearly, we can't allow z to lie on the real axis, because the integrand would have a pole on the path of integration. We could have made the restriction Im $z \neq 0$, but that (as we shall see) would give a discontinuous function — certainly not an analytic function, therefore.

The next question to answer is in what region of the complex plane does $F(z)$ define an analytic function. The integral exists in Im $z > 0$, and is in fact analytic there: the integrand is smooth; and convergence is uniform — though we need to integrate by parts to establish this, since convergence is a bit delicate.

Suppose that we want to analytically continue $F(z)$ to a neighbourhood of the point z_1, where Im $z_1 < 0$. We define a new function $F_1(z)$ by

$$F_1(z) = \int_\gamma \frac{e^{it}}{t-z} dt \qquad \text{for } z \text{ `above' } \gamma \text{ in the } z\text{-plane}$$

where the path of integration has been bent into the lower half t-plane such that the point $t = z_1$ lies 'above' γ as shown:

$F_1(z)$ is clearly analytic for any z lying 'above' γ, for the same reasons as $F(z)$ is analytic for z lying above the real axis. Furthermore, if Im $z > 0$ then $F(z) = F_1(z)$, since the two integrals agree; this can be shown by contour deformation: if Im $z > 0$, γ can be smoothly deformed onto the real axis without crossing any singularities of the integrand. Thus the two conditions for analytic continuation are satisfied and F_1 is an analytic continuation of F.

Now consider the function

$$G(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} dt \qquad \text{Im } z \neq 0$$

which differs from $F(z)$ in its domain of definition. Clearly, it agrees with $F(z)$ if Im $z > 0$.

However, if Im $z < 0$ (but z lies 'above' γ)

$$F_1(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} dt + 2\pi i e^{iz} = G(z_1) + 2\pi i e^{iz} \qquad \text{Im } z < 0$$

as can be seen by wrapping γ round the pole at $t = z_1$ (see diagram below). This means that $G(z)$ is discontinuous: $F_1(z)$ is certainly continuous (being analytic), so $G(z)$ jumps by $2\pi i e^{iz}$ as z crosses the real axis.

Note that we can evaluate $G(z)$ by closing the path of integration in the lower half plane and using the residue theorem: the result is 0 if z is in the upper half plane, but $2\pi i \exp(iz)$ if z is in the lower half plane, giving a discontinuous function as expected.