Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Solution of the Airy equation by integral representation

In this example, the Airy equation is solved using the Laplace representation. This equation is of exceptional importance both in its own right (for example, in optics) and as an approximation to the equation $w''+f(x)w=0$ near a point at which $f(x)=0$ (this is the basis of one treatment of the WKB approximation).

We obtain two linearly independent integral representation — as expected for a second-order differential equation. Since the integrand (the same in both cases) is entire and nowhere zero, the representations differ only in the directions in which the paths of integration go out to infinity.

We will find solutions of the Airy equation

$$w'' + zw = 0$$

in the form

$$w(z) = \int_\gamma f(t)e^{zt}dt \qquad (*)$$

where $f(t)$ is an as yet unknown function and γ is a path to be determined once $f(t)$ is known. We expect that there will be many choices of γ, two of which lead to linearly independent solutions of the differential equation. The function $f(t)$ will be uniquely determined up to (obviously) a multiplicative constant.

We start by substituting $(*)$ into the differential equation, differentating under the integral sign:

$$\int_\gamma (t^2 + z)f(t)e^{zt}dt = 0.$$

We cannot immediately say that the integral is identically zero so the integrand must vanish. However, we can determine circumstances in which the integrand will vanish, and if this gives two linearly independent functions, we are finished.

The immediate problem is the presence of the z term in the integral. We can't alleviate the problem by a clever choice of $f(t)$ involving z, because we assumed when differentiating under the integral sign that f did not depend on z. Our strategy, which will work for any polynomial in z, is to remove the z term by integrating by parts:

$$\begin{aligned}\int_\gamma zf(t)e^{zt}dt &= \int_\gamma f(t)\frac{\partial e^{zt}}{\partial t}dt \\ &= \int_\gamma \left(\frac{\partial\left(f(t)e^{zt}\right)}{\partial t} - \frac{df}{dt}e^{zt}\right)dt \\ &= f(t)e^{zt}\big|_\gamma - \int_\gamma \frac{df}{dt}e^{zt}dt,\end{aligned}$$

where the first term of the last equation is to be evaluated at the endpoints of the path γ.

Thus we require

$$f(t)e^{zt}\big|_\gamma + \int_\gamma \left(t^2 f(t) - f'(t)\right)e^{zt}dt = 0.$$

This we can achieve if we choose $f(t)$ such that

$$t^2 f(t) - f'(t = 0 \quad \text{i.e.} \quad f(t) = e^{\frac{1}{3}t^3},$$

where the constant of integration in $f(t)$ has been omitted since it corresponds merely to an constant multiple in $w(z)$, and if we choose γ such that

$$e^{\frac{1}{3}t^3+zt}\Big|_\gamma = 0. \qquad (**)$$

The obvious choice of γ which satisfies $(**)$ is a closed curve. This works; but since the function $f(t)e^{zt}$, which appears in the integrand of $w(z)$, is entire the corresponding solution to the Airy equation is $w(z) = 0$. The only other possibility, in view of the fact that $f(t)e^{zt} \neq 0$ for any value of t, is a path that begins and ends in sectors of $t = \infty$ for which $f(t)e^{zt} \to 0$ as $t \to \infty$. Note that the end points of γ cannot depend on the position of z, because (again) that would have given rise to extra terms when we differentiated the integral.

Now $e^{t^3+zt} \approx e^{t^3}$ for large t, so (setting $t = |t|e^{i\theta}$) we need to begin and end the path in sectors for which $\cos(3\theta) < 0$. There are three such sectors (see sketch) so any two of three paths will do; clearly, we can't use all three since the paths sum to a closed curve and hence the three corresponding solutions sum to zero.

What we haven't shown is that any two functions of the form

$$\int_\gamma e^{\frac{1}{3}t^3+zt} dt,$$

where $\gamma = \gamma_1$, γ_2 or γ_3, are linearly independent. The easiest way to do this is to consider asymptotic values, which is beyond the scope of this course. Alternatively, we could calculate $w(0)$ and $w'(0)$ for each solution (they come out in terms of Γ functions), and show that their ratios are different for the different functions.