Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Laplace Transforms: the inversion theorem

Let $f(t) = 0$ for $t < 0$ and assume that the Laplace transform of $f(t)$ exists — which means that $f(t)$ is integrable over the range $[0, R]$, for any R, and $f(t)e^{kt} \to 0$ as $t \to \infty$ for some k.

Then the inverse transform of $\widehat{f}(p)$ is

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{pt}\widehat{f}(p)dp$$

where the path of integration runs to the right of all singularities of the integrand. This integral is called the *Bromwich integral*.

Proof of inversion theorem

Note first that for $t < 0$ we can close the contour with a large semi-circle in the right half plane. Since by definition of the Bromwich integral the integrand has no singularities to the right of the path of integration, we have $f(t) = 0$ for $t < 0$, as required.

Of course, this is assuming that $\widehat{f}(p)$ does not tend to infinity exponentially in the right half plane; this can be shown to be the case (and is fairly clear: just consider the behaviour of the integrand of the Laplace transform as $p \to \infty$).

If the integrand is roughly e^{-pt_0} for large p in the right half plane, then for $t < t_0$ we can close in the right half plane and obtain the stronger result that $f(t) = 0$ for $t < t_0$.

Now we show that the value of the Bromwich integral is $f(t)$.

Starting with the right hand side, we have

$$\int_{a-i\infty}^{a+i\infty} e^{pt}\widehat{f}(p)dp = \int_{a-i\infty}^{a+i\infty} e^{pt}\left(\int_{-\infty}^{\infty} e^{-p\tau}f(\tau)d\tau\right)dp$$

(recall that $f(t) = 0$ for $t < 0$)

$$= ie^{at}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iy(t-\tau)-a\tau}f(\tau)d\tau dy$$

(setting $p = a + iy$)

$$= ie^{at}\int_{-\infty}^{\infty} e^{-a\tau}f(\tau)\left(\int_{-\infty}^{\infty} e^{iy(t-\tau)}dy\right)d\tau$$

(swapping the order of integration)

$$= 2\pi i e^{at}\int_{-\infty}^{\infty} e^{-a\tau}f(\tau)\delta(t-\tau)d\tau$$

$$= 2\pi i f(t)\ .$$

To justify the use of the δ function, note that for any continuous L_1 function g, we have:

$$\begin{aligned}
\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{iyu} g(u)du\right) dy &= \lim_{R\to\infty}\int_{-R}^{R}\left(\int_{-\infty}^{\infty} e^{iyu} g(u)du\right) dy \\
&= \lim_{R\to\infty}\int_{-\infty}^{\infty} g(u)\left(\int_{-R}^{R} e^{iyu} dy\right) du \\
&= \lim_{R\to\infty}\int_{-\infty}^{\infty} g(u)\left(\frac{2\sin uR}{u}\right) du \\
&= \lim_{R\to\infty}\int_{-\epsilon}^{\epsilon} g(u)\left(\frac{2\sin uR}{u}\right) du + O\left(\frac{1}{R}\right)
\end{aligned}$$

(by the Riemann-Lebesgue lemma)

$$= 2g(0)\lim_{R\to\infty}\int_{-\epsilon}^{\epsilon}\left(\frac{\sin uR}{u}\right) du + O\left(\frac{1}{R}\right)$$

(expanding $g(u)$ by Taylor series (with remainder) and using the Riemann-Lebesgue lemma)

$$= 2g(0)\lim_{R\to\infty}\int_{-R\epsilon}^{R\epsilon}\left(\frac{\sin v}{v}\right) dv + O\left(\frac{1}{R}\right)$$

(setting $v = Ru$)

$$= 2g(0)\int_{-\infty}^{\infty}\left(\frac{\sin v}{v}\right) dv = 2g(0)\times\pi.$$