Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Laplace Transforms: inversion and convolution theorems

These theorems are very important. The proofs run along similar lines to those for the Fourier transform, so it seemed sensible to consign them to a handout. Essentially, the Laplace transform is related to the Fourier transform by a rotation in the complex plane. The differences arise because Laplace transforms have exponential convergence (and so can be used for a function such as $f(t) = t$) and 'start at' $t = 0$ rather than $t = -\infty$.

Inversion Theorem

Let $f(t) = 0$ for $t < 0$ and assume that the Laplace transform of $f(t)$ exists — which means that $f(t)$ is integrable over the range $[0, R]$, for any R, and $f(t)e^{kt} \to 0$ as $t \to \infty$ for some k.

Then the inverse transform of $\widehat{f}(p)$ is

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{pt}\widehat{f}(p)dp$$

where the path of integration runs to the right of all singularities of the integrand. This integral is called the *Bromwich integral.*

Proof of inversion theorem

Note first that for $t < 0$ we can close the contour with a large semi-circle in the right half plane. Since by definition of the Bromwich integral the integrand has no singularities to the right of the path of integration, we have $f(t) = 0$ for $t < 0$, as required.

Of course, this is assuming that $\widehat{f}(p)$ does not tend to infinity exponentially in the right half plane; this can be shown to be the case (and is fairly clear: just consider the behaviour of the integrand of the Laplace transform as $p \to \infty$).

If the integrand is roughly e^{-pt_0} for large p in the right half plane, then for $t < t_0$ we can close in the right half plane and obtain the stronger result that $f(t) = 0$ for $t < t_0$.

Now we show that the value of the Bromwich integral is $f(t)$.

Starting with the right hand side, we have

$$\int_{a-i\infty}^{a+i\infty} e^{pt}\widehat{f}(p)dp \;=\; \int_{a-i\infty}^{a+i\infty} e^{pt}\left(\int_{-\infty}^{\infty} e^{-p\tau}f(\tau)d\tau\right)dp$$

(recall that $f(t) = 0$ for $t < 0$)

$$= \; ie^{at}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iy(t-\tau)-a\tau}f(\tau)d\tau dy$$

(setting $p = a + iy$)

$$= \; ie^{at}\int_{-\infty}^{\infty} e^{-a\tau}f(\tau)\left(\int_{-\infty}^{\infty} e^{iy(t-\tau)}dy\right)d\tau$$

(swapping the order of integration)

$$= \; 2\pi ie^{at}\int_{-\infty}^{\infty} e^{-a\tau}f(\tau)\delta(t-\tau)d\tau = 2\pi i f(t)\ .$$

The convolution theorem

Let f and g satisfy $f(t) = 0 = g(t)$ for $t < 0$, and have Laplace transforms $\widehat{f}$ and $\widehat{g}$. Let h be the convolution of f and g:

$$h(t) = \int_{-\infty}^{\infty} f(t-u)g(u)du = \int_0^t f(t-u)g(u)du.$$

The second equality follows from $f(t-u) = 0$ for $t - u < 0$, i.e. for $u > t$, and $g(u) = 0$ for $u < 0$. Then

$$\widehat{h}(p) = \widehat{f}(p)\widehat{g}(p).$$

Proof

$$\widehat{h}(p) = \int_0^{\infty} \left(\int_0^t f(t-u)g(u)du \right) e^{-pt} dt$$

$$= \int_0^{\infty} \int_0^t f(t-u)g(u)e^{-pt} dudt$$

$$= \int_0^{\infty} \int_u^{\infty} f(t-u)g(u)e^{-pt} dtdu \qquad \text{(changing the order of integratin: see below)}$$

$$= \int_0^{\infty} \int_0^{\infty} f(v)g(s)e^{-p(s+v)} dvds \qquad \text{(setting } t = u + v,\ s = u\text{; the Jacobian} = 1)$$

$$= \left(\int_0^{\infty} f(v)e^{-pv} dv \right) \left(\int_0^{\infty} g(s)e^{-ps} ds \right)$$

as required.

The change of the order of integration (Frobenius's theorem) can be understood by the following diagram: