Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Fourier Transforms

This handout covers the basic properties of Fourier Transforms. It should be mainly revision, except for the penultimate item, which is about transforms of functions that depend on three variables rather than the usual one. This is not difficult: you just transform with respect to each variable consecutively.

Notation

We write the Fourier transform of $\mathrm{f}\,(x)$ as $\widetilde{\mathrm{f}}\,(k)$ where k is the conjugate variable to x, and define

$$\widetilde{\mathrm{f}}\,(k) = \int_{-\infty}^{\infty} \mathrm{f}\,(x)e^{-ikx}dx.$$

Properties

1. Linearity:

$$\mathrm{g}\,(x) = a\,\mathrm{f}_1(x) + b\,\mathrm{f}_2(x) \implies \widetilde{\mathrm{g}}\,(k) = a\widetilde{\mathrm{f}}_1(k) + b\widetilde{\mathrm{f}}_2(k).$$

   This just follows from the linearity of the defining integrals.

2. Scaling:

$$\mathrm{g}\,(x) = \mathrm{f}\,(ax) \implies \widetilde{g}(k) = |a|^{-1}\widetilde{\mathrm{f}}\,(k/a).$$

   This comes from changing variable $x \to ax$. If $a < 0$, the limits change sign, whence the mod signs.

3. Shifting:

$$\mathrm{g}\,(x) = \mathrm{f}\,(x - x_0) \implies \widetilde{\mathrm{g}}\,(k) = e^{-ikx_0}\widetilde{\mathrm{f}}\,(k);$$
$$\mathrm{g}\,(x) = e^{ik_0x}\,\mathrm{f}\,(x) \implies \widetilde{\mathrm{g}}\,(k) = \widetilde{\mathrm{f}}\,(k - k_0).$$

   The first of these requires a change of variable in the defining integral $x \to x - x_0$. Note that these results are essentially 'conjugates' of each other.

4. Derivatives:

$$\mathrm{g}\,(x) = \frac{d\,\mathrm{f}}{dx} \implies \widetilde{g}(k) = ik\,\widetilde{\mathrm{f}}\,(k);$$
$$\mathrm{g}\,(x) = x\,\mathrm{f}\,(x) \implies \widetilde{\mathrm{g}}\,(k) = i\frac{d}{dk}\,\tilde{\mathrm{f}}\,(k)$$

   The first of these requires integration by parts. The second cannot be managed by integrating the left hand side; instead, start with the right hand side and differentiate under the integral sign.

5. Convolution:
$$\mathrm{h} = \mathrm{f} * \mathrm{g} \implies \widetilde{\mathrm{h}}(k) = \widetilde{\mathrm{f}}(k)\,\widetilde{\mathrm{g}}(k),$$
where
$$\mathrm{f} * \mathrm{g}(x) = \int_{-\infty}^{\infty} \mathrm{f}(u)\,\mathrm{g}(x-u)du.$$
To prove this, just take the Fourier transform of the convolution and use the shifting result from above:
$$\begin{aligned}\widetilde{\mathrm{h}}(k) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \mathrm{f}(u)\,\mathrm{g}(x-u)du\right)e^{-ikx}dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(u)\,\mathrm{g}(x-u)e^{-ikx}dxdu \\ &= \int_{-\infty}^{\infty} \mathrm{f}(u)e^{-iku}\,\widetilde{\mathrm{g}}(k)du = \widetilde{\mathrm{f}}(k)\,\widetilde{\mathrm{g}}(k)\end{aligned}$$

6. 3-dimensional transform:
$$\begin{aligned}\widetilde{\mathrm{f}}(\mathbf{k}) &= \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{-ik_1x}\,\mathrm{f}(\mathbf{x})dx\Big)e^{-ik_2y}dy\Big)e^{-ik_3z}dz \\ &= \int_{\text{all space}} e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{f}(\mathbf{x})d^3x = \int_{\text{all space}} e^{-ikr\cos\theta}\,\mathrm{f}(\mathbf{x})r^2\sin\theta dr\,d\theta\,d\phi.\end{aligned}$$
For the last expression, polar coordinates in **x**-space have been used. The polar direction (North pole) is taken to be parallel to **k**, so that θ is the angle between **k** (considered fixed) and **x**.

7. Transform of a constant (using the theory of distributions, giving a result that is consistent with the inversion theorem): $\widetilde{1} = 2\pi\delta(k)$.

Inversion Theorem

If $\int_{-\infty}^{\infty} |\,\mathrm{f}(x)|dx < \infty$ and $\mathrm{f}(x)$ is continuous then
$$\mathrm{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\tilde{\mathrm{f}}(k)dk.$$
This is the *Fourier representation* of f. If $\mathrm{f}(x)$ is only piecewise continuous, then at a discontinuity the Fourier representation gives the average value of f . The conditions can be weakened.

In three dimensions, we have
$$\mathrm{f}(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^3 \int_{\text{all }\mathbf{k}\text{-space}} e^{i\mathbf{k}\cdot\mathbf{x}}\,\tilde{\mathrm{f}}(\mathbf{k})d^3k,$$
which we might have to invert using polar coordinates in **k**-space, taking the polar direction to be parallel to **x**.