SOLAR SYSTEM

Exercise Sheet 1

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1.1 (40%)
Consider the study of the preference for commensurability in the solar system carried out by Roy & Ovenden (see Sect. 1.7). Pairs of integers, i_1 and i_2 with $i_1 < i_2 \leq i_{\max}$, can be used to generate N_r rationals of the form i_1/i_2. (a) By reducing all rationals to their simplest form, calculate the value of N_r for all values of $i_{\max}$ between 2 and 10 inclusive (b) If $\epsilon_{\max}$ is defined to be half the separation of the two closest of the N_r rationals, write down an expression for $\epsilon_{\max}$ in terms of $i_{\max}$. (c) Any ratio of orbital periods, T_1/T_2 (with $T_1 < T_2$), that lies in the permitted range $r_{\min} - \epsilon_{\max} \leq T_1/T_2 \leq r_{\max} + \epsilon_{\max}$ (where $r_{\min}$ and $r_{\max}$ are the smallest and largest of the N_r rationals) differs by a quantity $\epsilon = |T_1/T_2 - j/k|$ from the nearest rational, j/k, belonging to the set. Show that the probability, p, that a given ratio, T_1/T_2 is commensurable (i.e. has $\epsilon < \epsilon_{\max}$) is

$$p = \frac{2\epsilon_{\max} N_\mathrm{r}}{(i_{\max} - 2)/i_{\max} + 2\epsilon_{\max}}$$

(d) If N_p pairs of orbital periods in a system are found to lie within the permitted range and N_obs of these are observed to be commensurable within a tolerance $\epsilon_{\max}$, show that the probability, P, of this occurring by chance is

$$P = \frac{N_\mathrm{p}!}{(N_\mathrm{p} - N_\mathrm{obs})!\, N_\mathrm{obs}!} p^{N_\mathrm{obs}} (1-p)^{N_\mathrm{p} - N_\mathrm{obs}}.$$

1.2 (60%)
A test particle approaches a planet of mass M and radius R from infinity with speed v_∞ and an impact parameter p. Use the particle's energy and angular momentum with respect to the planet to derive expressions for the semi-major axis and eccentricity of the hyperbolic orbit followed by the test particle about M, and for the pericentre distance r_0. Show that the eccentricity may be written $e = 1 + 2v_\infty^2/v_0^2$ where v_0 is the escape velocity at r_0. Use the expression for the true anomaly corresponding to the asymptote of the hyperbola ($r \to \infty$) to show that the overall deflection of the test particle's orbit after it leaves the vicinity of the planet ψ, is given by $\sin(\psi/2) = e^{-1}$. Given that r_0 must be greater than R to avoid a physical collision, calculate the maximum deflection angles for (i) a spacecraft skimming Jupiter, with $v_\infty = 10$ km s^{-1}, and (ii) the *Cassini* orbiter skimming Saturn's large moon Titan, at $v_\infty = 5$ km s^{-1}.

C.D. Murray (14/10/10)