Mathematical Tripos Part IA — Lent term 2009

Dynamics and Relativity, Examples sheets — Dr S.T.C. Siklos

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Examples sheet 1

1 The coordinate systems (x,t) and (x',t') of two frames of reference S and S' respectively are related by

$$x' = f(x,t), \qquad t' = t$$

for some function f. A particle follows a trajectory $x = X(t)$ in S, and its trajectory in S' is $x' = X'(t') = f(X(t'),t')$. Using the chain rule, show that its speed u' and acceleration a' in S' are related to its speed u and acceleration a in S by

$$u' = u\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}, \qquad a' = a\frac{\partial f}{\partial x} + u^2\frac{\partial^2 f}{\partial x^2} + 2u\frac{\partial^2 f}{\partial x\,\partial t} + \frac{\partial^2 f}{\partial t^2}.$$

Given that S and S' are both inertial frames, use Newton's second law (with zero force) to show that $f(x,t) = bx + vt + c$ for some constants b, c and v.

2 A ball of mass m is projected vertically upwards with initial speed u in a resisting medium that produces a retardation force of magnitude kv^2, where v is the ball's speed. Show by means of dimensional analysis that when the ball returns to its point of projection, its speed w can be written in the form

$$w = uf(\lambda)\,,$$

where $\lambda = ku^2/mg$ (a dimensionless parameter). Integrate the equations of motion to show that $f(\lambda) = (1+\lambda)^{-1}$.

Discuss the cases (i) $\lambda \gg 1$, and (ii) $\lambda \ll 1$.

3 At time $t = 0$, an insect of mass m jumps from a point O on the ground with velocity $\mathbf{V}$, while a wind of velocity $\mathbf{U}$ is blowing. The gravitational acceleration is $\mathbf{g}$ and the air exerts a force on the insect equal to mk times the velocity of the wind *relative to the insect.*

(i) Show that the path of the insect is given by

$$\mathbf{r} = (\mathbf{U} + \mathbf{g}/k)t + k^{-1}(\mathbf{V} - \mathbf{U} - \mathbf{g}/k)(1 - \mathrm{e}^{-kt}).$$

(ii) In the case where the insect jumps vertically in a horizontal wind, show that the time T that elapses before it returns to earth satisfies

$$(1 - e^{-kT}) = kT(1+\epsilon)^{-1}, \qquad (*)$$

where $\epsilon = kV/g$, and find an expression for the range R in terms of ϵ, U and T. (Here $V = |\mathbf{V}|$, $g = |\mathbf{g}|$, and $U = |\mathbf{U}|$.)

(iii) When $\epsilon \ll 1$, expand both sides of $(*)$ and hence obtain the approximation

$$kT \approx 2\epsilon - \tfrac{2}{3}\epsilon^2.$$

(You will need the cubic term of the exponential.) Obtain also an approximation (two terms) for R in terms of U, k and ϵ

4 A satellite falls freely towards the Earth starting from rest at a distance R, much larger than the Earth's radius. Treating the Earth as a point of mass M, use dimensional analysis to show that the time T taken by the satellite to reach the Earth is given by

$$T = C\left(\frac{R^3}{GM}\right)^{\frac{1}{2}},$$

where G is the gravitational constant and C is a dimensionless constant.

By integrating the equation of motion of the satellite, show that $C = \pi/2\sqrt{2}$.

[The acceleration due to the Earth's gravitational field at a distance r from the centre of the Earth is GM/r^2.]

5 A particle of mass m and charge q moves in a constant uniform horizontal magnetic field $\mathbf{B}$ under the influence of gravity $\mathbf{g}$ acting vertically downwards. Show that the particle has a helical motion but with a constant horizontal drift, which you should find. An experimenter wishes to eliminate the drift by imposing a uniform electric field $\mathbf{E}$; what should be the direction and magnitude of $\mathbf{E}$?

6 A particle of mass m and charge $-e$ moves in a constant uniform magnetic field $\mathbf{B}$ towards a charge q fixed at the origin. The force on the particle is

$$\mathbf{F} = -\frac{qe}{4\pi\epsilon_0 r^3}\mathbf{r} - e\dot{\mathbf{r}} \times \mathbf{B},$$

where $\mathbf{r}(t)$ is its position and $r = |\mathbf{r}|$ (assumed to be non-zero). Show (by differentiating it) that the quantity

$$\frac{1}{2}m\dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{qe}{4\pi\epsilon_0 r}$$

is a constant, and give a physical interpretation of each term.

If initially $\mathbf{r}.\mathbf{B} = \dot{\mathbf{r}}.\mathbf{B} = 0$ explain, by considering the Taylor series of $\mathbf{r}.\mathbf{B}$ about $t = 0$ (which you may assume exists), why $\mathbf{r}.\mathbf{B} = 0$ at all later times. Show that, in this case, $m\mathbf{r} \times \dot{\mathbf{r}} - \frac{1}{2}er^2\mathbf{B}$ is also constant.

7 A particle at position vector $\mathbf{r}$ experiences a force $(ar^{-3} + br^{-4})\mathbf{r}$. Sketch the potential as a function of r in the different cases that arise according to the signs of a and b. If the particle starts a distance r_0 from the origin, find the minimum initial speed that must be imparted to it in order to escape to infinity in the different cases that arise according to the signs of a and b, and the magnitude of $r_0 a/b$.

[**NB** $\boldsymbol{\nabla} r = r^{-1}\mathbf{r}$.]

8 In a hypothetical universe there is a sun which exerts no gravitational force. Instead it is surrounded by a force field which gives any other particle an acceleration

$$\ddot{\mathbf{r}} = \lambda \mathbf{r} \times \dot{\mathbf{r}},$$

where $\mathbf{r}$ is the position vector of the particle relative to the sun at time t and λ is a constant. Show that particles move in this field with constant speeds. By considering $\mathbf{r}.\mathbf{r}$, or otherwise, show further that if the speed of the particle is v, and if it does not hit the sun, then its distance r from the sun is given by

$$r^2 = v^2\{(t - t_0)^2 + t_1^2\},$$

where t_0 and t_1 are constants.

9 The temperature $\theta(x, t)$ in a very long rod is governed by the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2},$$

where D is a constant (the *thermal diffusivity* of the rod). At time $t = 0$, the point $x = 0$ is heated to a high temperature. At all later times, the conservation of heat energy implies that

$$\int_{-\infty}^{\infty} \theta(x, t) dx = Q,$$

where Q is constant. Use dimensional analysis to show that $\theta(x, t)$ can be written in the form

$$\theta(x, t) = \frac{Q}{(Dt)^{\frac{1}{2}}} F(z),$$

where $z = x/(Dt)^{\frac{1}{2}}$ and show further that

$$\frac{d^2F}{dz^2} + \frac{z}{2}\frac{dF}{dz} + \frac{1}{2}F = 0.$$

Integrate this equation once directly to obtain a first order differential equation (you may assume that $F(z) \to 0$ and $dF(z)/dz \to 0$ as $z \to \infty$), and hence show that

$$\theta(x, t) = \frac{Q}{(4\pi Dt)^{\frac{1}{2}}} e^{-x^2/4Dt}.$$

Examples sheet 2

1 A rocket, moving vertically upwards, ejects gas vertically downwards at speed u relative to the rocket. Derive, giving careful explanations, the equation of motion

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - gm\,,$$

where v and m are the speed and total mass of the rocket (including fuel) at time t.

If u is constant and the rocket starts from rest with total mass m_0, show that

$$m = m_0 e^{-(gt+v)/u}.$$

2 A firework of initial mass m_0 is fired vertically upwards from the ground. The rate of burning of fuel $\mathrm{d}m/\mathrm{d}t$ is constant and equal to $-\alpha$, and the fuel is ejected at constant speed u relative to the firework. Show that the speed of the firework at time t, where $0 < t < m_0/\alpha$, is

$$v(t) = -gt - u\log\Big(1 - \frac{\alpha t}{m_0}\Big),$$

and that this is positive provided $u > m_0 g/\alpha$.

Suppose now that nearly all of the firework consists of fuel, the mass of the containing shell being negligible. Show that the height attained by the shell when all of the fuel is burnt is

$$\frac{m_0}{\alpha}\Big(u - \frac{m_0 g}{2\alpha}\Big).$$

3 A particle moves in a fixed plane and its position vector at time t is $\mathbf{r}$. Let (r,θ) be plane polar coordinates and let $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ be unit vectors in the direction of increasing r and increasing θ respectively. Show that the velocity and acceleration of the particle are given by

$$\dot{\mathbf{r}} = \dot r\,\hat{\boldsymbol{r}} + r\dot\theta\,\hat{\boldsymbol{\theta}}, \qquad \ddot{\mathbf{r}} = (\ddot r - r\dot\theta^2)\,\hat{\mathbf{r}} + (r\ddot\theta + 2\dot r\dot\theta)\,\hat{\boldsymbol{\theta}}.$$

(i) The particle moves with constant speed V on the equiangular spiral $r = a\exp(\theta\cot\alpha)$, where a and α are constants. Show that

$$V = r\dot\theta\,\mathrm{cosec}\,\alpha$$

and hence that

$$\dot{\mathbf{r}} = V\cos\alpha\,\hat{\mathbf{r}} + V\sin\alpha\,\hat{\boldsymbol{\theta}}.$$

Hence find the (plane polar) components $\ddot{\mathbf{r}}$ and show that $\dot{\mathbf{r}}.\ddot{\mathbf{r}} = 0$. Evaluate $|\ddot{\mathbf{r}}|$ in terms of r, V and α.

(ii) If, instead, the particle moves with variable speed $v(t)$ on the same spiral, but so that $\dot\theta$ takes a constant value ω, show that the acceleration has magnitude v^2/r and is directed at an angle 2α to the position vector.

4 A particle of unit mass moves with speed v in the gravitational field of the Sun and is influenced by radiation pressure. The forces acting on the particle are μ/r^2 towards the sun and kv opposing the motion, where μ and k are constants. Establish the equations

$$r^2\dot{\theta} = h\mathrm{e}^{-kt}, \qquad \mu r = h^2\mathrm{e}^{-2kt} - r^3(\ddot{r} + k\dot{r}),$$

where r and θ are plane polar coordinates centred on the Sun and h is a constant.

Show that when $k = 0$, a circular orbit of radius a exists for any value of a, and find its angular frequency ω in terms of a and μ.

When $k/\omega \ll 1$, the radius r changes very slowly, so that $\dot{r}$ and $\ddot{r}$ may be neglected compared to other terms. Verify that in this situation an approximate solution is

$$r = a\mathrm{e}^{-2kt}, \qquad \dot{\theta} = \omega\mathrm{e}^{3kt}.$$

Give a brief qualitative description of the behaviour of this solution for $t > 0$. Does the speed of the particle increase or decrease?

5 In an attractive $1/r$ potential, show that:
(i) for circular and parabolic orbits having the same angular momentum the perihelion distance r_{min} of the parabola is half the radius of the circle;
(ii) the speed of a particle at any point of a parabolic orbit is $\sqrt{2}$ times the speed in a circular orbit passing through the same point.

6 A particle A is attracted by the gravitational force of a second particle B which is fixed at the origin. Initially, A is very far from B and has velocity $\mathbf{V}$ directed along a straight line which passes at a distance l from B. The shortest distance between A's trajectory and B is d. Deduce the mass of B in terms of the given quantities and the gravitational constant G.

7 A particle P of unit mass moves in a plane under a central force

$$-\lambda u^3 - \mu u^2,$$

where $u = 1/r$ and λ, μ are positive constants. If P is projected with speed V from the point $r = r_0$, $\theta = 0$ in the direction perpendicular to OP, find the equation of the orbit under the assumptions

$$\lambda < V^2 r_0^2 < 2\mu r_0 + \lambda.$$

Explain the significance of these inequalities.

Show that between consecutive apsides (points of greatest or least distance) the radius vector turns through an angle

$$\pi\big(1 - \lambda/(V^2 r_0^2)\big)^{-1/2}.$$

Under what condition is the orbit a closed curve?

8 A particle P of mass m moves under the influence of a central force of magnitude mkr^{-3} directed towards a fixed point O. Initially $r = a$ and P has velocity V perpendicular to OP, where $V^2 < k/a^2$. Prove that P spirals in towards O and reaches O in a time

$$T = \frac{a^2}{\sqrt{k - a^2V^2}}.$$

9 A particle of unit mass moves in a central force field with potential

$$V(r) = -\frac{\alpha}{r},$$

where $\alpha > 0$. Show that the angular momentum $\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$ about the origin is constant, and deduce that the orbit lies in a plane containing the origin.

Let $\mathbf{M} = \dot{\mathbf{r}} \times \mathbf{L} - \alpha\mathbf{r}/r$. Show that $\mathrm{d}\mathbf{M}/\mathrm{d}t = \mathbf{0}$ and deduce that $\mathbf{M}$ is constant. By evaluating $\mathbf{M}$ at perihelion, show that $\mathbf{M}$ is directed along the major axis from the focus to the perihelion, and that $|\mathbf{M}| = \alpha e$ where e is the eccentricity of the orbit. **Hint**: use $r = (h^2/\alpha)/(1 + e\cos\theta)$.

Examples sheet 3

1 In a 2-dimensional spacetime an inertial frame $\mathcal{S}'$ moves with velocity u relative to an inertial frame $\mathcal{S}$. Write down an appropriate Lorentz transformation between $\mathcal{S}$ and $\mathcal{S}'$.

A particle p moves with speed v with respect to $\mathcal{S}$ and v' with respect to $\mathcal{S}'$, so that if its position is measured at two successive instants $dx = vdt$ and $dx' = v'dt'$. Suppose the two clocks agree for p, i.e., $dt' = dt$. Show that p is moving with constant velocity

$$v = \frac{c^2}{u}\left[1 - \sqrt{1 - u^2/c^2}\right].$$

2 A clock $\mathcal{C}$ is at rest at the spatial origin of an inertial frame $\mathcal{S}$. A second clock $\mathcal{C}'$ is at rest at the spatial origin of an inertial frame $\mathcal{S}'$ moving with constant speed u relative to $\mathcal{S}$. The clocks read $t = t' = 0$ when the two spatial origins coincide.

When $\mathcal{C}'$ reads t'_2 it receives a radio signal from $\mathcal{C}$ sent out when $\mathcal{C}$ reads t_1. Draw a spacetime diagram describing this process.

Determine the space-time coordinates (ct_2, x_2) in $\mathcal{S}$ of the point (event) at which $\mathcal{C}'$ receives the radio signal. Hence show that

$$t_1 = t'_2\sqrt{\frac{1 - u/c}{1 + u/c}}.$$

3 In an inertial frame $\mathcal{S}$ a photon with energy E moves in the xy-plane at an angle θ relative to the x-axis. Show that in a second frame $\mathcal{S}'$ whose relative speed is u directed in the x-direction, the energy and angle are given by

$$E' = \gamma E(1 - \beta \cos\theta), \qquad \cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta},$$

where $\beta = u/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. Write down E and $\cos\theta$ as functions of E' and $\cos\theta'$.

Show that for a photon moving in the x-direction there is a frequency change by a factor $\sqrt{(1-\beta)/(1+\beta)}$ — this is the *special relativistic Doppler effect.*

Next consider a source of photons which is at rest in $\mathcal{S}'$. Consider the photons emitted in the forward direction, i.e., $\cos\theta' > 0$. Show that if β is close to unity, these photons will appear in $\mathcal{S}$ to be concentrated in a narrow cone about $\theta = 0$ — this is the *headlight effect.*

4 Pulsars are stars which emit pulses of radiation at regular intervals. Jack and Jill count pulses from a very distant pulsar in the y direction. She travels at a speed given by $\beta = 24/25$ in the x-direction for seven years and then comes back at the same speed, while he stays at home. At the end of the trip they have counted the same number of pulses. Use question 3 to show that on return she has aged by 14 years and he by 50.

Obtain the corresponding result when the pulsar is in the x direction and draw a diagram to show why Jack and Jill count the same number of pulses.

5 A particle of rest mass m_0 disintegrates at rest into two particles of rest masses m_1 and m_2. Use conservation of relativistic energy and relativistic 3-momentum to show that the energies of the particles are given by

$$E_1 = c^2\,\frac{m_0^2 + m_1^2 - m_2^2}{2m_0}, \qquad E_2 = c^2\,\frac{m_0^2 + m_2^2 - m_1^2}{2m_0}.$$

Let P_0 be the 4-momentum of the original particle, and let and P_1 and P_2 be the 4-momenta of the product particles. Derive the above results by considering $(P_0 - P_1)\cdot(P_0 - P_1)$ and $(P_0 - P_2)\cdot(P_0 - P_2)$.

6 A particle with 4-momentum P is detected by an observer with four-velocity U. Write down the components of U in the rest frame of the observer. Write down the components of P in terms of the rest mass, relativistic 3-momentum and speed v of the detected particle.

Show that

$$\frac{v}{c} = \sqrt{1 - \frac{(P\cdot P)c^2}{(P\cdot U)^2}}.$$

7 A photon (of zero rest mass) collides with an electron of rest mass m which is initially at rest in the laboratory frame. Show that the angle θ by which the photon is deflected (measured in the laboratory frame) is related to the magnitudes p and q of its initial and final momenta by

$$2\sin^2\frac{1}{2}\theta = \frac{mc}{q} - \frac{mc}{p}.$$

8 In a laboratory frame a particle of rest mass m_1 has energy E_1, and a second particle of rest mass m_2 is at rest. By considering the scalar quantity $(P_1+P_2)\cdot(P_1+P_2)$, or otherwise, show the combined energy in the centre of momentum frame (i.e. the frame in which the total 3-momentum is zero) is

$$\sqrt{m_1^2c^2+m_2^2c^2+2E_1m_2}.$$

Hence show that in a collision of one proton with energy E on another one at rest it is possible to create a proton-antiproton pair (in addition to the original protons) if $E \geq 7m$ (where $m_1 = m_2 = m$ is the mass of a proton and of an antiproton).

9 A rocket ejects exhaust at constant speed u relative to itself by a process that conserves relativistic energy and momentum. Let m be the rest mass of the rocket when the rocket has speed v measured in the *initial* rest frame of the rocket. By equating the the 4-momentum of the rocket P with the total 4-momentum at a later time when the 4-momentum of the rocket is $P + dP$, show that

$$\frac{d(m\gamma v)}{dv} = \left(\frac{v-u}{1-uv/c^2}\right)\frac{d(m\gamma)}{dv},$$

where $\gamma = (1-v^2/c^2)^{-\frac{1}{2}}$. Rearrange this as a differential equation for $m\gamma$ and derive the relatistic rocket equation

$$v = c\tanh(\tfrac{u}{c}\log r)$$

where $r = m_0/m$ and m_0 is the initial rest mass of the rocket.

Now suppose that the rocket has an ideal photon drive, so that all matter to be ejected is converted first to photons by means of positron-electron annihilation. By considering directly initial and final energy and momentum, derive the relativistic rocket equation.

Examples Sheet 4

1 Coriolis force questions. Where required, use ω for the angular speed of the Earth and assume that events take place at latitude λ in the northern hemisphere.

(i) Are bath-plug vortices in the northern hemisphere likely, on average, to be clockwise or anticlockwise?

(ii) A straight river flows with speed v in a direction α degrees East of North. Show that the effect of the coriolis force is to undermine the right bank. Does the magnitude of the effect depend on α?

(iii) On a very calm day, the sea freezes. A particle is projected along the frozen surface. Ignoring the centrifugal force, show that the particle moves in a circle and determine the radius of the circle in terms of λ, the speed of the particle, and ω. Is the trajectory clockwise or anticloclwise? Why can the centrifugal force be ignored?

(iv) A plumb line is attached to the ceiling inside one of the carriages of a train and hangs down freely, at rest relative to the train. When the train is travelling at speed V in the north-easterly direction the plumb line hangs at an angle θ to the direction in which it hangs when the train is at rest. Ignoring centrifugal forces, show that $\theta \approx (2\omega V\sin\lambda)/g$. Why can the centrifugal force be ignored?

2 A square hoop $ABCD$ is made of fine smooth wire and has side length $2a$. The hoop is horizontal and rotating with constant angular speed ω about a vertical axis through A. A small bead which can slide on the wire is initially at rest at the midpoint of the side BC. Choose axes fixed relative to the hoop, and let y be the distance of the bead from the vertex B on the side BC. Write down the position vector of the bead in the rotating frame.

Using the standard expression for acceleration in a rotating frame, show that

$$\ddot{y} - \omega^2 y = 0.$$

Hence show that the time which the bead takes to reach a corner of the hoop is $\omega^{-1}\cosh^{-1} 2$.

3 A bullet of mass m is fired from a point $\mathbf{r}_0$ with velocity $\mathbf{u}$ in a frame which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame. The bullet is subject to a gravitational force $m\mathbf{g}$ which is constant in the rotating frame. Using the vector equation of motion and neglecting terms of order $|\boldsymbol{\omega}|^2$, show that the bullet's position vector measured in the rotating frame is approximately

$$\mathbf{r}_0 + \mathbf{u}t + (\frac{1}{2}\mathbf{g} - \boldsymbol{\omega} \times \mathbf{u})t^2 + \frac{1}{3}\mathbf{g} \times \boldsymbol{\omega} t^3$$

at time t.

Suppose that the bullet is projected from sea level on the Earth at latitude λ in the Northern hemisphere, at an angle $\pi/4$ from the upward vertical and in a Northward direction. Show that when the particle returns to sea level (neglecting the curvature of the Earth's surface), it has been deflected to the East by an amount approximately equal to

$$\frac{\sqrt{2}\,\omega|\mathbf{u}|^3}{3g^2}(3\sin\lambda - \cos\lambda),$$

where ω is the angular speed of the Earth.

Evaluate the approximate size of this deflection at latitude 52° N for $|\mathbf{u}| = 1000\,\mathrm{m/s}$.

4 A system of particles with masses m_i and position vectors $\mathbf{x}_i$, $i = 1, \ldots, n$, moves under its own mutual gravitational attraction alone. Show that a possible solution of the equations of motion is given by $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the $\mathbf{a}_i$ are constant vectors, if the $\mathbf{a}_i$ satisfy

$$\mathbf{a}_i = \frac{9G}{2}\sum_{j \neq i} \frac{m_j(\mathbf{a}_i - \mathbf{a}_j)}{|\mathbf{a}_i - \mathbf{a}_j|^3}.$$

Show that, for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other fixed point?

5 Two particles of masses m_1 and m_2 move under their mutual gravitational attraction. Show from first principles that the quanitity

$$\tfrac{1}{2}\dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{GM}{r}$$

is constant, where $\mathbf{r}$ is the position vector of one particle relative to the other and $M = m_1+m_2$.

The particles are released from rest a long way apart, and fall towards each other. Show that the position of their centre of gravity is fixed, and that when they are a distance r apart their relative speed is $\sqrt{2GM/r}$.

When the particles are a distance a apart, They are given equal and opposite impulses, each of magnitude I, and each perpendicular to the direction of motion. Show that subsequently $r^2\omega = aI/\mu$, where ω is the angular speed of either particle relative to the centre of mass and μ is the reduced mass of the system.

Show further that the minimum separation, d, of the two particles in the subsequent motion satisfies

$$(a^2 - d^2)I^2 = 2GM\mu^2 d\ .$$

6 State the parallel axis and perpendicular axis theorems.

Find the moment of inertia of a uniform solid circular cone of mass M, height h and base radius a about its axis, and also about a perpendicular axis through its apex.

7 (i) Thin circular discs of radius a and b are made of uniform materials with mass per unit area ρ_a and ρ_b, respectively. They lie in the same plane. Their centres A and B are connected by a light rigid rod of length c. Find the moment of inertia of the system about an axis through B perpendicular to the plane of the discs.

(ii) A thin uniform circular disc of radius a and centre A has a circular hole cut in it of radius b and centre B, where $AB = c < a - b$. The disc is free to oscillate in a vertical plane about a smooth fixed horizontal circular rod of radius b passing through the hole. Using the result of part (i), with ρ_b suitably chosen, show that the period of small oscillations is $2\pi\sqrt{l/g}$, where $l = c + (a^4 - b^4)/(2a^2c)$.

8 A yo-yo consists of two uniform discs, each of mass M and radius R, connected by a short light axle of radius a around which a portion of a thin string is wound. One end of the string is attached to the axle and the other to a fixed point P. The yo-yo is held with its centre of mass vertically below P and then released.

Assuming that the unwound part of the string remains approximately vertical, use the principle of conservation of energy to find the equation of motion of the centre of mass of the yo-yo. What is the tension in the string when the centre of mass has fallen a distance y?

9 A particle of mass m is attached to the end B of a light rod AB of length h. Calculate the inertia tensor about the point A using axes such that (i) the z-axis is parallel to the rod and (ii) the rod lies in the x-z plane inclined at an angle θ to the z axis.