Chapter 1

Dimensional Analysis

The first step in modeling any physical phenomena is the identification of the relevant variables, and then relating these variables via known physical laws. For sufficiently simple phenomena we can usually construct a quantitative relationship among these variables from first principles; however, for many complex phenomena (which often occur in engineering applications) such an *ab initio* theory is often difficult, if not impossible. In these situations modeling methods are indispensable, and one of the most powerful modeling methods is dimensional analysis. You have probably encountered dimensional analysis in your previous physics courses when you were admonished to "check your units" to ensure that the left and right hand sides of an equation had the same units (so that your calculation of a force had the units of kg m/s^2). In a sense, this is all there is to dimensional analysis, although "checking units" is certainly the most trivial example of dimensional analysis (incidentally, if you aren't in the habit of checking units, do it!). Here we will use dimensional analysis to actually *solve* problems, or at least infer some information about the solution. Much of this material is taken from Refs. [1] and [2]; Ref. [3] provides many interesting applications of dimensional analysis and scaling to biological systems (the science of *allometry*).

The basic idea is the following: *physical laws do not depend upon arbitrariness in the choice of the basic units of measurement.* In other words, Newton's second law, $\mathbf{F} = m\mathbf{a}$, is true whether we choose to measure mass in kilograms, acceleration in meters per second squared, and force in newtons, or whether we measure mass in slugs, acceleration in feet per second squared, and force in pounds. As a concrete example, consider the angular frequency of small oscillations of a point pendulum of length l and mass m:

$$\omega = \sqrt{\frac{g}{l}}, \tag{1.1}$$

where g is the acceleration due to gravity, which is 9.8 m/s^2 on earth (in the SI system of units; see below). To derive Eq. (1.1), one usually needs to solve the differential equation which results from applying Newton's second law to the pendulum (do it!). Let's instead deduce (1.1) from dimensional considerations alone. What can ω depend upon? It is reasonable to assume that the relevant variables are m, l, and g (it is hard to imagine others, at least for a point pendulum). Now suppose that we change the system of units so that the unit of

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mass is changed by a factor of M, the unit of length is changed by a factor of L, and the unit of time is changed by a factor of T. With this change of units, the units of frequency will change by a factor of T^{-1}, the units of velocity will change by a factor of LT^{-1}, and the units of acceleration by a factor of LT^{-2}. Therefore, the units of the quantity g/l will change by T^{-2}, and those of $(g/l)^{1/2}$ will change by T^{-1}. Consequently, the ratio

$$\Pi = \frac{\omega}{\sqrt{g/l}} \tag{1.2}$$

is invariant under a change of units; Π is called a *dimensionless number*. Since it doesn't depend upon the variables (m, g, l), it is in fact a constant. Therefore, from dimensional considerations alone we find that

$$\omega = \text{const.} \times \sqrt{\frac{g}{l}}. \tag{1.3}$$

A few comments are in order: (1) the frequency is independent of the mass of the pendulum bob, a somewhat surprising conclusion to the uninitiated; (2) the constant cannot be determined from dimensional analysis alone. These results are typical of dimensional analysis—uncovering often unexpected relations among the variables, while at the same time failing to pin down numerical constants. Indeed, to fix the numerical constants we need a real *theory* of the phenomena in question, which goes beyond dimensional considerations.

1.1 Units

Before proceeding further with dimensional considerations we first need to discuss units of measurement.

1.1.1 The SI system of units

In this course we will adopt the SI system of units,[1] which is described in some detail in the *Physicist's Desk Reference* [4] (which I will abbreviate as PDR from now on), pp. 4–10. In the SI system the base, or *defined*, units, are the meter (m), the kilogram (kg), the second (s), the kelvin (K), and the ampere (A).[2] The definitions of these units in terms of fundamental physical processes are given in the PDR. All other units are *derived*. For instance, the SI unit of energy, the joule (J), is equal to 1 kg m^2/s^2. The derived units are also listed in the PDR. The SI system is referred to as a LMT-class, since the defined units are length L, mass M, and time T (if we add thermal and electrical phenomena, then we have a $LMT\theta I$-class in the SI system).

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[1]In some older texts this is referred to as the *MKS* system.

[2]We should also add the mole (mol) and the candela (cd), but these will seldom enter into our models.

1.1.2 Atomic units

The SI system of units is not the only possible choice, however, and often the choice of units is dictated by the scale of the physical phenomena under consideration. For instance, in studying phenomena at the atomic and molecular level it is useful to choose a unit of length which is comparable to the size of an atom. We do this by using the electron charge e as the unit of charge, and the electron mass m_e as the unit of mass. At the atomic level the forces are all electromagnetic, and therefore the energies are always proportional to $e^2/4\pi\epsilon_0$, which has dimensions ML^3T^{-2}. Another quantity that appears in quantum physics is $\hbar$, Planck's constant divided by 2π, which has dimensions ML^2T^{-1}. Dimensional analysis then tells us that the atomic unit of length is

$$a_B = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)} = 0.529 \times 10^{-10} \text{ m.} \tag{1.4}$$

This is called the *Bohr radius*, or simply the bohr, because in the Bohr model it is the radius of the smallest orbit for an electron circling a fixed proton.

One can similarly find the unit of time by dimensional analysis. Rather than do this directly, we find first the atomic unit of energy, which is

$$E_h = \frac{e^2}{4\pi\epsilon_0}\frac{1}{a_B} = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m_e}{\hbar^2} = 4.36 \times 10^{-18} \text{ J} = 27.2 \text{ eV,} \tag{1.5}$$

where eV="electron volt"=1.6×10^{-19} J; this is the *kinetic energy* acquired by an electron which is accelerated from rest through a potential difference of one Volt. The energy E_h is called the *hartree*, and is twice the ground state energy of an electron circling a fixed proton. The unit of time is then $\hbar/E_h$.

It is worth noting that in atomic units the unit of velocity is $a_B/(\hbar/E_h) = e^2/(4\pi\epsilon_0\hbar)$; the ratio of this velocity to the speed of light is a dimensionless number called the *fine structure constant* α:

$$\alpha = \frac{(e^2/4\pi\epsilon_0)}{\hbar c} \approx \frac{1}{137}. \tag{1.6}$$

The fact that this number is small is important when treating the interaction of radiation with matter.

1.2 Dimensions, dimensional homogeneity, and independent dimensions

Returning to the discussion above, recall that if the units of length are changed by a factor of L, and the units of time are changed by a factor of T, then the units of velocity change by a factor of LT^{-1}. We call LT^{-1} the *dimensions* of the velocity; it tells us the factor by which the numerical value of the velocity changes under a change in the units (within the LMT-class). Following a convention suggested by Maxwell, we denote the dimensions of a

Table 1.1: Dimensions of some commonly encountered physical quantities in the $LMT\theta I$-class.

$[L]$	Length	L
$[M]$	Mass	M
$[T]$	Time	T
$[v]$	Velocity	LT^{-1}
$[a]$	Acceleration	LT^{-2}
$[F]$	Force	MLT^{-2}
$[\rho]$	Mass density	ML^{-3}
$[p]$	Pressure	$ML^{-1}T^{-2}$
$[\alpha]$	Angle	1
$[E]$	Energy	ML^2T^{-2}
$[\theta]$	Temperature	θ
$[S]$	Entropy	$ML^2T^{-2}\theta^{-1}$
$[I]$	Electric current	I
$[Q]$	Electric charge	IT
$[E]$	Electric field	$MLT^{-3}I^{-1}$
$[B]$	Magnetic field	$MT^{-2}I^{-1}$

physical quantity ϕ by $[\phi]$; thus, $[v] = LT^{-1}$. A dimensionless quantity would have $[\phi] = 1$; i.e., its numerical value is the same in all systems of units within a given class. What about more complicated quantities such as force? From Newton's second law, $F = ma$, so that $[F] = [m][a] = MLT^{-2}$. Proceeding in this way, we can easily construct the dimensions of any physical quantity; some of the more commonly encountered quantities are included in Table 1.1.

We see that all of the dimensions in the Table are *power law monomials*, of the form (in the LMT-class)

$$[\phi] = CL^aM^bT^c, \tag{1.7}$$

where C and (a, b, c) are constants. In fact, this is a general result which can be proven mathematically; see Sec. 1.4 of Barenblatt's book, Ref. [1]. This property is often called *dimensional homogeneity*, and is really the key to dimensional analysis. To see why this is useful, consider again the determination of the period of a point pendulum, in a more abstract form. We have for the dimensions $[\omega] = T^{-1}$, $[g] = LT^{-2}$, $[l] = L$, and $[m] = M$. If ω is a function of (g, l, m), then its dimensions must be a power-law monomial of the dimensions of these quantities. We then have

$$\begin{aligned} [\omega] &= T^{-1} \\ &= [g]^a[l]^b[m]^c \\ &= (LT^{-2})^a L^b M^c \\ &= L^{a+b}T^{-2a}M^c, \end{aligned} \tag{1.8}$$

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with a, b, and c constants which are determined by comparing the dimensions on both sides of the equation. We see that

$$a + b = 0, \quad -2a = -1, \quad c = 0. \tag{1.9}$$

The solution is then $a = 1/2$, $b = -1/2$, $c = 0$, and we recover Eq. (1.2).

A set of quantities $(a_1, \ldots, a_k)$ is said to have *independent dimensions* if none of these quantities have dimensions which can be represented as a product of powers of the dimensions of the remaining quantities. As an example, the density ($[\rho] = ML^{-3}$), the velocity ($[v] = LT^{-1}$), and the force ($[F] = MLT^{-2}$) have independent dimensions, so that there is no product of powers of these quantities which is dimensionless.[3] On the other hand, the density, velocity, and pressure ($[p] = ML^{-1}T^{-2}$) are not independent, for we can write $[p] = [\rho][v]^2$; i.e., $p/(\rho v^2)$ is a dimensionless quantity.

Now suppose we have a relationship between a quantity a which is being determined in some experiment (which we will refer to as the *governed parameter*), and a set of quantities $(a_1, \ldots, a_n)$ which are under experimental control (the *governing parameters*), which is of the form

$$a = f(a_1, \ldots, a_k, a_{k+1}, \ldots a_n), \tag{1.10}$$

where $(a_1, \ldots, a_k)$ have independent dimensions. For example, this would mean that the dimensions of the governed parameter a is determined by the dimensions of $(a_1, \ldots, a_k)$, while all of the a_s's with $s > k$ can be written as products of powers of of the dimensions of $(a_1, \ldots, a_k)$; e.g., $a_{k+1}/a_1^p \cdots a_k^r$, would be dimensionless, with $(p, \ldots, r)$ an appropriately chosen set of constants. With this set of definitions, it is possible to prove that Eq. (1.10) can be written as

$$a = a_1^p \cdots a_k^r \Phi\left(\frac{a_{k+1}}{a_1^{p_{k+1}} \cdots a_k^{r_{k+1}}}, \ldots, \frac{a_n}{a_1^{p_n} \cdots a_k^{r_n}}\right), \tag{1.11}$$

with Φ some function of *dimensionless quantities only*. The great simplification is that while the function f in Eq. (1.10) was a function of n variables, the function Φ in Eq. (1.11) *is only a function of $n - k$ variables*. Eq. (1.11) is a mathematical statement of *Buckingham's Π-Theorem*, which is the central result of dimensional analysis. The formal proof can again be found in Barenblatt's book [1]. Dimensional analysis cannot supply us with the dimensionless function Φ—we need a real theory for that.

As a simple example of how this works, let's return to the pendulum, but this time we'll assume that the mass can be distributed, so that we relax the condition of the mass being concentrated at a point. The governed parameter is the frequency ω; the governing parameters are g, l (which we can interpret as the distance between the pivot point and the center of mass), m, and the moment of inertia about the pivot point, I. Since $[I] = ML^2$, the set (g, m, l, I) is not independent; we can choose as our independent parameters (g, m, l) as before, with I/ml^2 a dimensionless parameter. In the notation developed above, $n = 4$

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[3]Prove this formally by writing $[F]^a[v]^b[\rho]^c = 1$, and then show that the only solution is $a = b = c = 0$. Alternatively, show that it is impossible to write $[F] = [\rho]^a[v]^b$ for any a, b.

and $k = 3$. Therefore, dimensional analysis tells us that

$$\omega = \sqrt{\frac{g}{l}} \Phi \left(\frac{I}{ml^2} \right), \tag{1.12}$$

with Φ some function which cannot be determined from dimensional analysis alone; we need a *theory* in order to determine it.

1.3   Examples

Some of the preceding discussion may have seemed a little abstract; I will try to flesh it out a bit with several examples.

1.3.1   Oscillations of a star

A star undergoes some mode of oscillation. How does the frequency ω of oscillation depend upon the properties of the star? The first step is the identification of the physically relevant variables. Certainly the density ρ and the radius R are important; we'll also need the gravitational constant G which appears in Newton's law of universal gravitation. We could add the mass m to the list, but if we assume that the density is constant as a first approximation, then $m = \rho(4\pi R^3/3)$, and the mass is redundant. Therefore, ω is the governed parameter, with dimensions $[\omega] = T^{-1}$, and (ρ, R, G) are the governing parameters, with dimensions $[\rho] = ML^{-3}$, $[R] = L$, and $[G] = M^{-1}L^3T^{-2}$ (check the last one). You can easily check that (ρ, R, G) have independent dimensions; therefore, $n = 3, k = 3$, so the function Φ is simply a constant in this case. Next, determine the exponents:

$$\begin{aligned} [\omega] &= T^{-1} \\ &= [\rho]^a [R]^b [G]^c \\ &= M^{a-c} L^{-3a+b+3c} T^{-2c}. \end{aligned} \tag{1.13}$$

Equating exponents on both sides, we have

$$a - c = 0, \qquad -3a + b + 3c = 0, \qquad -2c = -1. \tag{1.14}$$

Solving, we find $a = c = 1/2$, $b = 0$, so that

$$\omega = C\sqrt{G\rho}, \tag{1.15}$$

with C a constant. We see that the frequency of oscillation is proportional to the square root of the density, and independent of the radius. Once again, the determination of C requires a real theory of stellar oscillation, but the interesting dependence upon the physical parameters has been obtained from dimensional considerations alone.

1.3.2 Gravity waves on water

Next consider waves on the surface of water, which are called *gravity waves* (or sometimes *capillary waves*). How does the frequency ω depend upon the wavenumber[4] k of the wave? The relationship $\omega = \omega(k)$ is known as the *dispersion relation* for the wave. The relevant variables would appear to be (ρ, g, k), which have dimensions $[\rho] = ML^{-3}$, $[g] = LT^{-2}$ and $[k] = L^{-1}$; these quantities have independent dimensions, so $n = 3$, $k = 3$. Now we can determine the exponents:

$$
\begin{aligned}
[\omega] &= T^{-1} \\
&= [\rho]^a [g]^b [k]^c \\
&= M^a L^{-3a+b-c} T^{-2b},
\end{aligned}
\tag{1.16}
$$

so that

$$
a = 0, \qquad -3a + b - c = 0, \qquad -2b = -1,
\tag{1.17}
$$

with the solution $a = 0$, $b = c = 1/2$. Therefore,

$$
\omega = C\sqrt{gk},
\tag{1.18}
$$

with C another undetermined constant. We see that the frequency of water waves is proportional to the square root of the wavenumber, in contrast to sound or light waves, for which the frequency is proportional to the wavenumber. This has the interesting consequence that the group velocity of these waves is $v_g = \partial\omega/\partial k = (C/2)\sqrt{g/k}$, while the phase velocity is $v_\phi = \omega/k = C\sqrt{g/k}$, so that $v_g = v_\phi/2$. Recall that the group velocity describes the large scale "lumps" which would occur when we superimpose two waves, while the phase velocity describes the short scale "wavelets" inside the lumps. For water waves these wavelets travel twice as fast as the lumps.

You might worry about the effects of surface tension σ on the dispersion relationship. We can include these in our dimensional analysis by recalling that the surface tension is the energy per unit area of the surface of the water, so it has dimensions $[\sigma] = MT^{-2}$. The dimensions of the surface tension are not independent of the dimensions of (ρ, g, k); in fact, it is easy to show that $[\sigma] = [\rho][g][k]^{-2}$, so that $\sigma k^2/\rho g$ is dimensionless. Then using the same arguments as before, we have

$$
\omega = \sqrt{gk}\,\Phi\left(\frac{\sigma k^2}{\rho g}\right),
\tag{1.19}
$$

with Φ some undetermined function. A calculation of the dispersion relation for gravity waves starting from the fundamental equations of fluid mechanics [5] gives

$$
\omega = \sqrt{gk}\sqrt{1 + \sigma k^2/\rho g},
\tag{1.20}
$$

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[4]Recall that $k = 2\pi/\lambda$, with λ the wavelength of the wave.

so that our function $\Phi(x)$ is

$$\Phi(x) = \sqrt{1 + x}. \tag{1.21}$$

Dimensional analysis enabled us to deduce the correct form of the solution, i.e., the possible combinations of the variables. Of course, only a complete theory could provide us with the function $\Phi(x)$.

What have we gained? We originally started with ω being a function of the four variables (ρ, g, k, σ); what dimensional analysis tells us is that it is really only a function of the combination $\sigma k^2/\rho g$, even though we don't know the function. Notice that this is an important fact if you are trying to measure the dependence of ω on the physical parameters (ρ, g, k, σ). If you needed to make (say) 10 separate measurements on each variable while holding the others fixed, then without dimensional analysis you would naively need to make 10^4 separate measurements. Dimensional analysis tells you that you only really need to measure the combinations gk and $\sigma k^2/\rho g$, so only need to make 10^2 measurements to characterize ω. Dimensional analysis can be a labor-saving device!

1.3.3   Energy in a nuclear explosion

We next turn to a famous example worked out by the eminent British fluid dynamicist G. I. Taylor.[5] In a nuclear explosion there is an essentially instantaneous release of energy E in a small region of space. This produces a spherical shock wave, with the pressure inside the shock wave thousands of times greater that the initial air pressure, which may be neglected. How does the radius R of this shock wave grow with time t? The relevant governing variables are E, t, and the initial air density ρ_0, with dimensions $[E] = ML^2T^{-2}$, $[t] = T$, and $[\rho_0] = ML^{-3}$. This set of variables has independent dimensions, so $n = 3$, $k = 3$. We next determine the exponents:

$$
\begin{aligned}
[R] &= L \\
&= [E]^a[\rho_0]^b[t]^c \\
&= M^{a+b}L^{2a-3b}T^{-2a+c}
\end{aligned}
\tag{1.22}
$$

so that

$$a + b = 0, \qquad 2a - 3b = 1, \qquad -2a + c = 0, \tag{1.23}$$

with the solution $a = 1/5$, $b = -1/5$, $c = 2/5$. Therefore

$$R = CE^{1/5}\rho_0^{-1/5}t^{2/5}, \tag{1.24}$$

with C an undetermined constant. If we could plot the radius R of the shock as a function of time t on a log-log plot, the slope of the line should be 2/5. The intercept of the graph would provide information about the energy E released in the explosion, if the constant C could be determined. By solving a model shock-wave problem Taylor estimated C to be

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[5]Taylor's name is associated with many phenomena in fluid mechanics: the Rayleigh-Taylor instability, Saffman-Taylor fingering, Taylor cells, Taylor columns, etc.

about 1; he was able to take de-classified movies of nuclear tests, and using his model, infer the yield of the bombs [6]. This data, of course, was strictly classified; it came as a surprise to the American intelligence community that this data was essentially publicly available to those well versed in dimensional analysis.

1.3.4 Solution of the diffusion equation

Dimensional analysis can also be used to solve certain types of partial differential equations. If this seems too good to be true, it isn't. Here we will concentrate on the solution of the diffusion equation. This material is optional, but you might find it interesting if you have some background in differential equations.

We'll start by deriving the one-dimensional diffusion, or heat, equation.[6] Let $\tau(x,t)$ represent the temperature of a metal bar at a point x at time t (I'll use τ to avoid confusion with the symbol for the dimension of time, T). The first step is the derivation of a *continuity equation* for the heat flow in the bar. Let the bar have a cross sectional area A, so that the infinitesimal volume of the bar between x and $x+\Delta x$ is $A\,\Delta x$. The quantity of heat contained in this volume is $c_p\tau A\,\Delta x$, with c_p the specific heat at constant pressure *per unit volume*; it has dimensions $[c_p] = ML^{-1}T^{-2}\theta^{-1}$. In a time interval dt this heat changes by an amount $c_p(\partial\tau/\partial t)A\,\Delta x dt$ due to the change in temperature. This change in the heat must come from somewhere, and is the result of a flux of heat $q(x,t)$ through the area A (q is the heat flowing through a unit area per unit time). Into the left side of the volume an amount of heat $qAdt$ flows in a time dt; on the right hand side of the volume a quantity $A[q+(\partial q/\partial x)\Delta x]dt$ flows out in a time dt, so that the net accumulation of heat in the volume is $-A(\partial q/\partial x)\Delta x dt$. Equating the two expressions for the rate of change of the heat in the volume $A\,\Delta x$, we find

$$c_p\frac{\partial\tau}{\partial t} = -\frac{\partial q}{\partial x}, \tag{1.25}$$

which is the equation of continuity. It is a mathematical expression of the conservation of heat in the infinitesimal volume $A\,\Delta x$. We supplement this with a phenomenological law of heat conduction, known as *Fourier's law*: the heat flux is proportional to the negative of the local temperature gradient (heat flows from a hot reservoir to a cold reservoir):

$$q = -\kappa\frac{\partial\tau}{\partial x}, \tag{1.26}$$

with κ the *thermal conductivity* of the metal bar. The thermal conductivity is usually measured in units of W cm^{-1} K^{-1}, and has dimensions $[\kappa] = MLT^{-3}\theta^{-1}$. See the Table on p. 38 of the PDR for the thermal conductivities of some materials. Combining Eqs. (1.25) and (1.26), we obtain the *diffusion equation* (often called the *heat equation*)

$$\frac{\partial\tau}{\partial t} = D\frac{\partial^2\tau}{\partial x^2}, \tag{1.27}$$

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[6]I prefer the term diffusion equation, since we are just describing the diffusion of heat.

where $D = \kappa/c_p$ is the *thermal diffusivity* of the metal bar; it has dimensions $[D] = [\kappa]/[c_p] = L^2T^{-1}$, as it should. Eq. (1.27) is the diffusion equation for heat. The diffusion equation will appear in many other contexts during this course. It usually results from combining a continuity equation with an empirical law which expresses a current or flux in terms of some local gradient.

Suppose that the bar is very long, so that we can consider the idealized case of an infinite bar. At an initial time $t = 0$, we add an amount of heat Q_0 (with dimensions $[Q_0] = ML^2T^{-2}$) at some point of the bar, which we will arbitrarily call $x = 0$. We could do this, for instance, by briefly holding a match to the bar. The heat is conserved at all times, so that

$$c_p A \int_{-\infty}^{\infty} \tau(x,t)dx = Q_0. \tag{1.28}$$

How does this heat diffuse away from $x = 0$ as a function of time t; i.e., what is $\tau(x,t)$? We first identify the important parameters. The temperature τ certainly depends upon x, t, and the diffusivity D; we see from Eq. (1.28) that it also depends upon the initial conditions through the combination $Q \equiv Q_0/c_p A$. What are the dimensions? We have $[x] = L$, $[t] = T$, $[D] = L^2T^{-1}$, and $[Q] = L\theta$, so that $n = 4$. These dimensions are not independent, for the quantity $x/(Dt)^{1/2}$ is dimensionless, so that $k = 3$. We will choose as our independent quantities (t, D, Q). Now express τ in terms of these variables:

$$\begin{aligned} [\tau] &= \theta \\ &= [t]^a [D]^b [Q]^c \\ &= L^{2b+c} T^{a-b} \theta^c. \end{aligned} \tag{1.29}$$

We find

$$2b + c = 0, \qquad a - b = 0, \qquad c = 1, \tag{1.30}$$

which has the solution $a = -1/2$, $b = -1/2$, $c = 1$. Therefore, dimensional analysis tells us that the solution of the diffusion equation is of the form

$$\tau(x,t) = \frac{Q}{(Dt)^{1/2}} \Phi\left(\frac{x}{(Dt)^{1/2}}\right), \tag{1.31}$$

with Φ a function which we still need to determine. The important point is that Φ is only a function of the combination $x/(Dt)^{1/2}$, and not x and t separately. To determine Φ, let's introduce the dimensionless variable $z = x/(Dt)^{1/2}$. Now use the chain rule to calculate various derivatives of τ:

$$\begin{aligned} \frac{\partial \tau}{\partial x} &= \frac{Q}{(Dt)^{1/2}} \frac{\partial z}{\partial x} \frac{d\Phi(z)}{dz} \\ &= \frac{Q}{Dt} \frac{d\Phi(z)}{dz}, \end{aligned} \tag{1.32}$$

$$\frac{\partial^2 \tau}{\partial x^2} = \frac{Q}{(Dt)^{3/2}} \frac{d^2\Phi(z)}{dz^2}, \tag{1.33}$$

$$\frac{\partial \tau}{\partial t} = -\frac{1}{2}\frac{Q}{D^{1/2}t^{3/2}}\Phi(z) + \frac{Q}{(Dt)^{1/2}}\frac{\partial z}{\partial t}\frac{d\Phi(z)}{dz}$$

$$= -\frac{1}{2}\frac{Q}{D^{1/2}t^{3/2}}\left[\Phi(z) + z\frac{d\Phi(z)}{dz}\right]. \qquad (1.34)$$

Substituting Eqs. (1.33) and (1.34) into the diffusion equation (1.27), and canceling various factors, we obtain a differential equation for Φ,

$$\frac{d^2\Phi(z)}{dz^2} + \frac{z}{2}\frac{d\Phi(z)}{dz} + \frac{1}{2}\Phi(z) = 0. \qquad (1.35)$$

Dimensional analysis has reduced the problem from the solution of a partial differential equation in two variables to the solution of an ordinary differential equation in one variable! The normalization condition, Eq. (1.28), becomes in these variables

$$\int_{-\infty}^{\infty} \Phi(z)\,dz = 1. \qquad (1.36)$$

You might think that Eq. (1.35) is hard to solve; however, it turns out that it is an exact differential,

$$\frac{d}{dz}\left[\frac{d\Phi}{dz} + \frac{z}{2}\Phi\right] = 0, \qquad (1.37)$$

which we can integrate once to obtain

$$\frac{d\Phi}{dz} + \frac{z}{2}\Phi = \text{const.} \qquad (1.38)$$

However, since any physically reasonable solution would have both $\Phi \to 0$ and $d\Phi/dx \to 0$ as $x \to \infty$, the integration constant must be zero. We now need to solve a first order differential equation, which we do by dividing Eq. (1.38) by Φ, multiplying by dz, and integrating, with the result that $\ln \Phi = -z^2/4 + \text{const.}$, or

$$\Phi(z) = Ce^{-z^2/4}, \qquad (1.39)$$

with C a constant. To determine C, we use the normalization condition, Eq, (1.36):

$$C\int_{-\infty}^{\infty} e^{-z^2/4}dz = C(4\pi)^{1/2} = 1, \qquad (1.40)$$

where the integral (known as a *Gaussian integral*) can be found in integral tables. Therefore $C = 1/(4\pi)^{1/2}$. Returning to our original variables, we have

$$\tau(x,t) = \frac{Q}{(4\pi Dt)^{1/2}}e^{-x^2/4Dt}. \qquad (1.41)$$

This is the complete solution for the temperature distribution in a one-dimensional bar due to a point source of heat.[7]

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[7]For the mathematically sophisticated, I'll mention that the same solution can be obtained using the method of Fourier transforms applied to the diffusion equation.

1.4 Similarity, modeling, and estimating

The notion of *similarity* is familiar from geometry. Two triangles are said to be similar if all of their angles are equal, even if the sides of the two triangles are of different lengths. The two triangles have the same shape; the larger one is simply a scaled up version of the smaller one. This notion can be generalized to include physical phenomena. This is important when *modeling* physical phenomena—for instance, testing a prototype of a plane with a scale model in a wind tunnel. The design of the model is dictated by dimensional analysis.

Return to the mathematical statement of the Π-Theorem, Eq. (1.11). We can identify the following dimensionless parameters:

$$\Pi = \frac{a}{a_1^p \cdots a_k^r}, \quad \Pi_1 = \frac{a_{k+1}}{a_1^{p_k+1} \cdots a_k^{r_k+1}}, \tag{1.42}$$

and so on, such that Eq. (1.11) can be written as

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_{n-k}). \tag{1.43}$$

The parameters $(\Pi, \Pi_1, \ldots, \Pi_{n-k})$ are known as *similarity parameters*. Now if two physical phenomena are similar, they will be described by the same function Φ. Denote the similarity parameters of the model and the prototype by the superscripts m and p, respectively. Then if the two are similar, their similarity parameters are equal:

$$\Pi_1^{(p)} = \Pi_1^{(m)}, \ldots, \Pi_{n-k}^{(p)} = \Pi_{n-k}^{(m)}, \tag{1.44}$$

so that

$$\Pi^{(p)} = \Pi^{(m)}. \tag{1.45}$$

Therefore, in order to have an accurate physical model of a prototype, we must first identify all of the similarity parameters, and then insure that they are equal for the model and the prototype.

Finally, we come to estimating. In this course we will often make *order of magnitude estimates*, where we try to obtain an estimate to within a factor of ten (sometimes better, sometimes worse). This means that we often drop factors of two, etc., although one should exercise some caution in doing this. Estimating in this fashion is often aided by first doing some dimensional analysis. Once we know how the governed parameter (which we are trying to estimate) scales with other quantities, we can often use our own personal experience as a guide in making the estimate. More examples of this later in the course (especially *elastic similarity*).

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