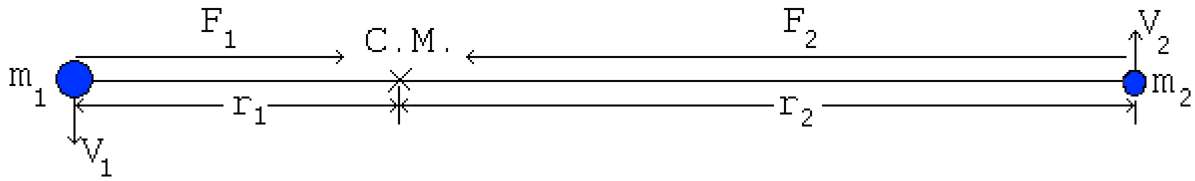
Derivation of Newton's form of Kepler's Third Law



$$F_{12}=F_{21}=\frac{Gm_1m_2}{a^2} \quad (1) \qquad \text{and} \qquad a=r_1+r_2$$

Consider the case of circular motion (which simplifies the maths compared to elliptical motion). Equation (1) represents Newton's Law of Universal Gravitation: the

two masses pull each other with equal and opposite forces F_{12} and F_{21}. These forces

are equal to the centripetal forces F_1 and F_2 keeping the masses orbiting their common centre of mass (CM). The CM is where the moments m_1r_1 and m_2r_2 are equal; it is the same as the centre of balance.

The Centripetal force is given by:

$$F_1=\frac{m_1v_1^2}{r_1} \quad (2) \qquad \text{and} \qquad F_2=\frac{m_2v_2^2}{r_2} \quad (3)$$

The expressions $\frac{v^2}{r}$ are formulae for accelerations in circular motion.

For circular motion, velocity is given by:

$$v_1=\frac{2\pi r_1}{P} \quad (4)$$ (Where *P* is the period of the circular motion)

Substitute (4) into (2) to get:

$$F_1=\frac{m_1}{r_1}\left(\frac{2\pi r_1}{P}\right)^2=\frac{4\pi^2m_1r_1}{P^2} \quad (5)$$

and an expression similar to equation (4) for m_2 may be substituted into (3):

$$F_2=\frac{m_2}{r_2}\left(\frac{2\pi r_2}{P}\right)^2=\frac{4\pi^2m_2r_2}{P^2} \quad (6)$$

But, according to Newton's first law (1),

$$F_1 = F_2 = \frac{G m_1 m_2}{a^2} = F_{12} = F_{21}$$

We saw above that the moments *mr* are equal, so $\frac{r_1}{r_2} = \frac{m_2}{m_1}$ and $r_2 = a - r_1$

(from the diagram), which gives

$$r_1 = r_2 \frac{m_2}{m_1} = (a - r_1)\frac{m_2}{m_1} = \frac{a m_2}{m_1} - \frac{r_1 m_2}{m_1} \quad (7)$$

Solve this to get:

$$r_1\left(1 + \frac{m_2}{m_1}\right) = \frac{a m_2}{m_1} \quad (8)$$

Multiply by m_1 to get:

$$r_1(m_1 + m_2) = a m_2 \quad (9)$$

and hence:

$$r_1 = a \frac{m_2}{(m_1 + m_2)} \quad (10)$$

From (5):

$$F_1 = \frac{4\pi^2 m_1 r_1}{P^2} = \frac{G m_1 m_2}{a^2} \quad (11)$$

Substitute for r_1 from (10):

$$\frac{4\pi^2}{P^2} a \left(\frac{m_2}{m_1 + m_2}\right) = \frac{G m_2}{a^2} \quad (12)$$

And re-arrange to make P^2 the subject:

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} = k a^3 \quad (13)$$ (where *k* is a constant)