Extrasolar Planets and Astrophysical Discs
Problem Set 2: Solutions

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The L_1 point lies between the two stars, so $x > x_2$ and $x < x_1$ around the L_1 point. We can use this fact to eliminate the modulus signs, giving

$$\Phi = -\frac{Gm_1}{x_1 - x} - \frac{Gm_2}{x - x_2} - \frac{1}{2}\Omega^2 x^2$$

on the x axis. Differentiating this,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = -\frac{Gm_1}{(x_1 - x)^2} + \frac{Gm_2}{(x - x_2)^2} - \Omega^2 x \ .$$

At the L_1 point, $\mathrm{d}\Phi/\mathrm{d}x = 0$, and substituting for $\Omega^2 = G(m_1 + m_2)/D^3$,

$$-\frac{Gm_1}{(x_1 - x)^2} + \frac{Gm_2}{(x - x_2)^2} - \frac{G(m_1 + m_2)}{D^3} x = 0$$

Substituting the expressions for x_1 and x_2 in terms of D, m_1 and m_2,

$$-\frac{G\,m_1}{\left(x - \frac{m_2 D}{m_1+m_2}\right)^2} + \frac{G\,m_2}{\left(x + \frac{m_1 D}{m_1+m_2}\right)^2} - \frac{G\,(m_1 + m_2)}{D^3} x = 0 \ ,$$

the required result.
Using $x = r + x_2 = r - m_1 D/(m_1 + m_2)$, the denominators in the above become

$$\left(x - \frac{m_2 D}{(m_1 + m_2)} \right) = r - D \qquad \text{and} \qquad \left(x + \frac{m_1 D}{(m_1 + m_2)} \right) = r \ .$$

Substituting these into the expression above,

$$-\frac{G\,m_1}{(r - D)^2} + \frac{G\,m_2}{r^2} - \frac{G\,(m_1 + m_2)}{D^3} \left(r - \frac{m_1 D}{(m_1 + m_2)} \right) = 0 \ ,$$

Therefore,

$$-\frac{G\,m_1}{(r - D)^2} + \frac{G\,m_2}{r^2} - \frac{G\,(m_1 + m_2)}{D^3} r + \frac{G\,m_1}{D^2} = 0 \ ,$$

the required result.
For small m_2, $m_2 \ll m_1$, so $m_1 + m_2 \simeq m_1$. For small r, $r/D \ll 1$. Therefore,

$$\frac{G\,m_1}{(r - D)^2} = \frac{G\,m_1}{(D - r)^2} = \frac{G\,m_1}{D^2 \left(1 - \frac{r}{D}\right)^2} = \frac{G\,m_1}{D^2} \left(1 - \frac{r}{D}\right)^{-2} \simeq \frac{G\,m_1}{D^2} \left[1 + \frac{2r}{D} - \frac{3r^2}{D^2}\right]$$

using the binomial theorem. Substituting this into the equation above and cancelling we obtain,

$$-\frac{2G\,m_1\,r}{D^3} + \frac{3G\,m_1 r^2}{D^4} + \frac{G\,m_2}{r^2} - \frac{G\,(m_1+m_2)}{D^3}\,r = 0$$

If we use $(m_1 + m_2) \simeq m_1$,

$$-\frac{3G\,m_1\,r}{D^3} + \frac{3G\,m_1 r^2}{D^4} + \frac{G\,m_2}{r^2} = 0$$

Rearranging,

$$\frac{3G\,m_1\,r}{D^3}\left(1 - \frac{r}{D}\right) = \frac{G\,m_2}{r^2}$$

But $r \ll D$, so $1 - \frac{r}{D} \simeq 1$. This gives,

$$\frac{3G\,m_1\,r}{D^3} = \frac{G\,m_2}{r^2}$$

which gives

$$r = D\left(\frac{m_2}{3\,m_1}\right)^{\frac{1}{3}} ,$$

the required result.