Extrasolar Planets and Astrophysical Discs
Problem Set 1: Solutions

January 2010

The potential energy is $E_g = -\frac{1}{2}G\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V\mathrm{d}V'$, where $\mathrm{d}V$ is an element of volume at position $\mathbf{r}$, $\mathrm{d}V'$ is an element of volume at position $\mathbf{r}'$ and V is the total volume of the cloud. But D is the maximum distance between two points. Therefore, $|\mathbf{r}-\mathbf{r}'| \leq D$. So,

$$E_g < -\frac{1}{2}G\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{D}\mathrm{d}V\mathrm{d}V' = -\frac{1}{2}\frac{G}{D}\int_V\int_V \rho(\mathbf{r})\rho(\mathbf{r}')\mathrm{d}V\mathrm{d}V' = -\frac{1}{2}\frac{G}{D}M^2$$

because $\int_V\int_V \rho(\mathbf{r})\rho(\mathbf{r}')\,\mathrm{d}V\mathrm{d}V' = \left(\int_V \rho(\mathbf{r})\,\mathrm{d}V\right)\left(\int_V \rho(\mathbf{r}')\,\mathrm{d}V'\right) = (M)\,(M) = M^2$. The maximum magnetic flux density is B_0. Therefore $B^2 \leq B_0^2$. So the energy in the magnetic field is

$$\mathcal{M} < \int_V \frac{B_0^2}{2\mu_0}\,\mathrm{d}V = \frac{B_0^2}{2\mu_0}\int_V \mathrm{d}V = \frac{B_0^2}{2\mu_0}V \;.$$

Note that the inequality ($<$ rather than $\leq$) comes from the fact that B_0 is the maximum value of B and that $B < B_0$ in some places.
The maximum dimension of the cloud is D, so $V \leq \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 < \frac{4}{3}\pi D^3$, whatever the configuration of the cloud. Therefore,

$$\mathcal{M} < \frac{4\pi D^3}{3}\frac{B_0^2}{2\mu_0} \;.$$

The cloud is at rest, and assuming that there is no internal motion, $\mathcal{K} = 0$. Putting the limits on E_g and $\mathcal{M}$ into the virial theorem equation, we get

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} < 2\left(\frac{4\pi D^3}{3}\frac{B_0^2}{2\mu_0}\right) + 4\,(0) + 2\left(-\frac{1}{2}\frac{G\,M^2}{D}\right)$$

which gives the required result, $\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} < -\frac{GM^2}{D} + \frac{4\pi D^3 B_0^2}{3\mu_0}$.
Using the $-\frac{GM^2}{D} + \frac{4\pi D^3 B_0^2}{3\mu_0} < 0$, condition given in the question, we get

$$\pi^2 D^4 B_0^2 < \frac{3\pi\mu_0 G M^2}{4}$$

(note that the physical quantities G, D, B_0, μ_0 are all positive).
Define the parameter $\mathcal{F}$ as $\mathcal{F} \equiv \pi D^2 B_0$, therefore $\pi^2 D^4 B_0^2 = \mathcal{F}^2 < \frac{3\pi\mu_0 G M^2}{4}$.
Since D is the largest dimension across the cloud, $\pi D^2 >$ largest area in the cloud.

We have magnetic flux = area × flux density at a point. So $\pi D^2 B_0 > F_m$, the maximum magnetic flux through any surface in the cloud (with $\mathcal{F}$ and $F_m > 0$). But $\mathcal{F} \equiv \pi D^2 B_0$. Therefore, $\mathcal{F} > F_m$, the required result. So $F_m^2 < \mathcal{F}^2$ and $F_m^2 < \dfrac{3\pi\mu_0 GM^2}{4}$, the required result.

For a magnetic field to start to inhibit the collapse, we expect $\pi^2 D^4 B_0^2 \simeq \dfrac{3\pi\mu_0 GM^2}{4}$, instead of the inequality. Therefore, $B_0 \simeq \sqrt{\dfrac{3\mu_0 GM^2}{4\pi D^4}} \simeq 9 \times 10^{-11}\ \mathrm{T} \simeq 1\mu\mathrm{G}$ using the given parameters.
(Note that the Gauss, G, is an old unit of magnetic flux density. The Gausss is related to the S.I. unit the Tesla, T, by $1\ \mathrm{G} \equiv 10^{-4}\ \mathrm{T}$.)

Flux conservation gives $\pi D_0^2 B_0 = \pi D_f^2 B_f$, where $D_0 = 10^{16}$ m is the initial size and $D_f = 7 \times 10^8$ m and B_f are the final size and flux density.

Therefore, $B_f = \left(\dfrac{D_0}{D_f}\right)^2 B_0 \simeq \left(\dfrac{10^{16}}{10^8}\right)^2 \times 8.9 \times 10^{-11}\ \mathrm{T} \simeq 1.8 \times 10^4\ \mathrm{T}$.
So, if the cloud did collapse to a size $1R_\odot$, the field would be about 2×10^4 T.