King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3630 General Relativity and Cosmology

Summer 2004

Time Allowed: THREE Hours

Candidates should answer no more than SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	$\epsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}
Speed of light in free space	$c = 2.998 \times 10^8$ m s^{-1}
Gravitational constant	$G_\mathrm{N} = 6.673 \times 10^{-11}$ N m^2 kg^{-1}
Elementary charge	$e = 1.602 \times 10^{-19}$ C
Electron rest mass	$m_\mathrm{e} = 9.109 \times 10^{-31}$ kg
Unified atomic mass unit	$m_\mathrm{u} = 1.661 \times 10^{-27}$ kg
Proton rest mass	$m_\mathrm{p} = 1.673 \times 10^{-27}$ kg
Neutron rest mass	$m_\mathrm{n} = 1.675 \times 10^{-27}$ kg
Planck constant	$h = 6.626 \times 10^{-34}$ J s
Boltzmann constant	$k_\mathrm{B} = 1.381 \times 10^{-23}$ J K^{-1}
Stefan-Boltzmann constant	$\sigma = 5.670 \times 10^{-8}$ W m^{-2} K^{-4}
Gas constant	$R = 8.314$ J mol^{-1} K^{-1}
Avogadro constant	$N_\mathrm{A} = 6.022 \times 10^{23}$ mol^{-1}
Molar volume of ideal gas at STP	$= 2.241 \times 10^{-2}$ m^3
One standard atmosphere	$P_0 = 1.013 \times 10^5$ N m^{-2}
Schwarzschild metric (SM) (in units with $G_N = c = 1$)	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + +r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$
Shell coordinates in SM:	$dt_\mathrm{shell} = (1 - \frac{2M}{r})^{1/2} dt \; ; \quad dr_\mathrm{shell} = (1 - \frac{2M}{r})^{-1/2} dr$
Christoffel symbols:	$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$.
Riemann Curvature Tensor (RCT):	$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\kappa\mu}\Gamma^\kappa{}_{\beta\nu} - \Gamma^\alpha{}_{\kappa\nu}\Gamma^\kappa{}_{\beta\mu}$.
Properties of RCT:	$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$.
Ricci tensor:	$R_{\mu\nu} = R_{\nu\mu} = R^\alpha{}_{\mu\alpha\nu}$.
Cosmic Horizon in Friedmann-Robertson-Walker Universe:	$\delta(t) = a(t) \int_{t_0}^{\infty} \frac{dt'}{a(t')}$.

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SECTION A - Answer SIX parts of this section

1.1) Consider the effective potential of the Schwarzschild solution:

$$\left(\frac{V_{\rm eff}(r)}{m}\right)^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{(L/m)^2}{r^2}\right)$$

where the symbols have their usual meaning. Discuss qualitatively the main features of the motion of a satellite of mass m in this potential. Compare these with the Newtonian case, stressing the main differences.

[7 marks]

1.2) Consider a space time whose Ricci tensor has the form $R_{\mu\nu} = A g_{\mu\nu}$, where the constant $A > 0$, and $g_{\mu\nu}$ is the metric tensor. Show that this space time is an exact solution of Einstein equations without matter but with a cosmological constant, and determine this constant in the case of four space-time dimensions.

[7 marks]

1.3) Wien's law of thermodynamics states that the maximum of thermal radiation spectrum has a wavelength $\lambda_{\rm max}$ which changes with the temperature $T_{\rm rad}$ of radiation according to: $\lambda_{\rm max} T_{\rm rad} = \text{constant}$. Moreover, the thermal radiation satisfies the law of Black-Body radiation according to which its energy, and hence its mass density, $\rho_{\rm rad}$, scales with the temperature as:

$$\rho_{\rm rad} = \alpha \frac{T_{\rm rad}^4}{c^2}$$

where α is the radiation constant. Using the cosmological redshift, which you should state without proof, and the above expression, show that

$$\rho_{\rm rad} \propto a^{-4} \ ,$$

where a is the scale factor of a Robertson-Walker Universe.

[7 marks]

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1.4) Consider an ideal fluid in a four-dimensional Minkowski space time with metric $\eta_{\mu\nu} = \mathrm{Diag}(-1, 1, 1, 1)$.

(i) Write down the conservation law of energy and momentum in terms of the stress tensor $T_{\mu\nu}$ of the fluid in a covariant form. Consider the stress tensor $T_{\mu\nu} = -|B|^2\eta_{\mu\nu}$, where B is a constant. Explain whether or not this satisfies the conservation law.

(ii) how is the conservation law modified in the case of a general metric $g_{\mu\nu}$? in that case, does the tensor $T^{\rm gen}_{\mu\nu} = -|B|^2 g_{\mu\nu}$ satisfy the conservation law and why?

[7 marks]

1.5) Explain qualitatively how the theory of Big-Bang accounts for the fact that the sky is dark at night.

[7 marks]

1.6) Consider two observers who are static with respect to each other as well as to the Earth. Light at frequency ν is emitted by one observer, and received at frequency ν' by the other who lies at a height H directly above the first, in the gravitational field of the Earth. The gravitational redshift implies that there is a change in frequency $\nu - \nu'$ between the emission (ν) and reception (ν') points, given by:

$$\frac{\nu' - \nu}{\nu} = -\frac{gH}{c^2} \ ,$$

where g is the acceleration of gravity, and the height H is assumed relatively small, so that g is approximately constant. Using appropriate space-time diagrams, explain briefly how the above phenomenon cannot be compatible with special relativity.

[7 marks]

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1.7) Which of the following expressions represent a proper invariant line element in general relativity, and why?

(i) $A_1 = -dx^2 + x(dy^2 + dz^2) + dw$,

(ii) $A_2 = -dx^2 + x(dy^2 + dz^2) + dydz$.

(iii) $A_3 = -dt^2 + dx^3 + dy^2 + dz^2$.

In the case of the proper invariant line element write down the components of the metric tensor.

[7 marks]

1.8) Explain briefly why the exterior of a spherically symmetric pulsating star cannot support gravitational waves. Does the exterior of a collapsing binary star system support gravitational waves in principle? Justify briefly your answer.

[7 marks]

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SECTION B - Answer TWO questions

2) Consider the two-dimensional spacetime described by the infinitesimal line element:

$$ds^2 = -dt^2 + a(t)^2 dr^2,$$

where t is the time coordinate.

(i) What does this space-time represent?

[2 marks]

(ii) By using an appropriate variational method, or otherwise, compute the Christoffel symbols for the above spacetime.

[6 marks]

(iii) Compute the independent components of the Riemann tensor in this two dimensional geometry.

[6 marks]

(iv) Show that the non-vanishing components of the Ricci tensor, for this spacetime are:

$$R_{tt} = -\frac{1}{a(t)} \frac{d^2 a(t)}{dt^2}, \qquad R_{rr} = a(t) \frac{d^2 a(t)}{dt^2}.$$

[8 marks]

(v) Compute the curvature scalar of this spacetime, and discuss the evolution in cosmic time for the case $a^2(t) = t$. What do you conclude on the existence of a cosmic horizon? Discuss the behaviour of the universe in the two limiting cases $t \to \infty$, and $t = 0$.

[8 marks]

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3) Light in General Relativity follows, by definition, null geodesics. Consider a three-dimensional space time with Schwarzschild geometry, that is, assume $d\phi = 0$ and $\theta \in [0, 2\pi]$ in the respective formulae in the rubric.

(i) Consider *radial* motion of light in this three-dimensional Schwarzschild space time. Work in units for which $G_N = c = 1$. Show that the radial velocity of light in book-keeper Schwarzschild coordinates is given by

$$\frac{dr}{dt} = \pm \left(1 - \frac{2M}{r}\right) \ .$$

Explain the physical meaning of the $\pm$ sign in this formula.

[8 marks]

(ii) Carry out a similar analysis as in (i) but for *tangential* motion of light in the three-dimensional Schwarzschild geometry, and show that the tangential velocity

$$r\frac{d\theta}{dt} = \pm \left(1 - \frac{2M}{r}\right)^{1/2} \ .$$

[8 marks]

(iii) Explain why the results in (i) and (ii) do not contradict the special theory of relativity.

[6 marks]

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(iv) The general radial part of the equations of motion for light in book-keeper coordinates of this geometry can be shown to be:

$$\frac{dr}{dt} = \pm(1 - \frac{2M}{r}) \left(1 - (1 - \frac{2M}{r})\frac{b^2}{r^2}\right)^{1/2} \quad ,$$

where $b = L/E$ is the impact parameter, with L the angular momentum, and E the total energy of the light. Use shell coordinates (see rubric) to show that:

$$\frac{1}{b^2}\left(\frac{dr_{\text{shell}}}{dt_{\text{shell}}}\right)^2 = \frac{1}{b^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}$$

[2 marks]

(v) From the result of (iv) define the effective potential for light as:

$$V_{\text{light}}^2 = \frac{1 - \frac{2M}{r}}{r^2}$$

What can you conclude from this expression concerning the dependence of V_{light} on the photon wavelength ?

[1 mark]

Sketch the function V_{light}^2 versus r/M.

[3 marks]

What is your conclusion regarding the possibility of having stable circular orbits of light ?

[2 marks]

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4) Consider an expanding Universe described by a Robertson-Walker space time:

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{1}$$

where χ is the radial coordinate, $a(t)$ the scale factor and $f(\chi) = \sin\chi$ for closed, $f(\chi) = \chi$ for flat, and $f(\chi) = \sinh\chi$ for open Universe.

(i) Relate the coordinate χ to the distance d from a bright celestial object as measured by an observer at rest with respect to the coordinate system of (1), and show that $d = a(t)\chi$.

[7 marks]

(ii) State Hubble's law and use the results of (i) to prove it.

[7 marks]

(iii) Consider the radial motion of light in the space time (1). Using a method of your choice, write down the geodesics corresponding to the radial χ coordinate, and show that they take form $dp_\chi/d\lambda = 0$, where λ is the affine parameter, and p_χ is the canonical momentum corresponding to the radial coordinate χ. Thus, p_χ = constant, which by normalization can be set to $p_\chi = -1$, where the minus sign is due to the fact that the direction of the photon is towards the observer.

[7 marks]

(iv) Given that the photon - viewed as a particle - is massless, show from (iii) that the covariant four-momentum of the photon can be written as: $p_\mu = (-\frac{1}{a(t)}, -1, 0, 0)$.

[3 marks]

(v) Use without proof that the frequency ν of a photon with a covariant four-momentum p_μ, as measured by an observer who moves with a four-velocity u^μ with respect to the cosmological frame, is $\nu = -p_\mu u^\mu$. Show that $\nu a(t)$ = constant, in the case of a photon emitted by an observer who is static with respect to the cosmological frame, and received by another observer who is also static with respect to that frame.

[6 marks]

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5) **(i)** For an expanding Universe with scale factor $a(t)$, regarded as an ideal fluid, the change in the total energy dE satisfies the thermodynamic relation: $dE = -pdV$, where dV denotes the change in the proper (spatial) volume, and p is the pressure. Show that for a fluid with equation of state $p = w\rho$, with w a constant, and ρ the mass density, one obtains:

$$a\frac{d\rho}{da} + 3(1+w)\rho = 0 \tag{2}$$

[8 marks]

(ii) Integrate (2) to obtain the scaling law of ρ as a function of $a(t)$, that is:

$$\rho \sim a^{-3(1+w)} \ .$$

[8 marks]

(iii) Consider the two cases of 'dust' and 'pure radiation'. Using the result of (ii) for the two cases separately compute the respective scaling laws for the mass densities, $\rho_{\rm dust}$ and $\rho_{\rm rad}$ respectively, and then show that $\rho_{\rm rad}/\rho_{\rm dust} \propto 1/a(t)$.

[8 marks]

(iv) Using the Stefan-Boltzmann law for the energy of thermal radiation, $E_{\rm rad} = \alpha T^4$, where α is the radiation constant, and the fact that energy is equivalent to mass multiplied by c^2 in relativity (where c is the speed of light in vacuo), show that the temperature of a radiation-dominated Universe is inversely proportional to the scale factor.

[6 marks]

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