King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3630 General Relativity and Cosmology

Summer 2003

Time Allowed: THREE Hours

Candidates must answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Speed of light in free space	$c = 2.998 \times 10^8$ meters $(\text{sec})^{-1}$
Newton constant	$G_N = 6.673 \times 10^{-11}$ $(\text{meters})^3$ $(\text{sec})^{-2}$ kg^{-1}
Planck constant	$h = 6.626 \times 10^{-34}$ J sec
Mass of Earth	$M_\oplus = 5.974 \times 10^{24}$ kg
	$= 4.44 \times 10^{-3}$ meters
Radius of Earth	$r_\oplus = 6.37 \times 10^6$ meters
Mass of Sun	$M_\odot = 1.989 \times 10^{30}$ kg
	$= 1.477 \times 10^3$ meters
Schwarzschild metric (SM)	$ds^2 = -d\tau^2 = -\left(1 - \frac{2M}{r}\right) dt^2 +$
(in units with $G_N = c = 1$)	$\left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$
Energy in Schwarzschild Geometry (SG):	$\frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$
Angular Momentum in SG	$\frac{L}{m} = r^2 \frac{d\phi}{d\tau}$
Effective Potential in SG (satellite motion):	$\frac{\mathcal{U}(r)}{m} \equiv \frac{1}{2}(\frac{E}{m})^2 - \frac{1}{2}(\frac{dr}{d\tau})^2$
Christoffel symbols:	$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$.
Riemann Tensor (RT):	$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\kappa\mu}\Gamma^\kappa{}_{\beta\nu} - \Gamma^\alpha{}_{\kappa\nu}\Gamma^\kappa{}_{\beta\mu}$.
Properties of RT:	$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$.
Ricci tensor:	$R_{\mu\nu} = R_{\nu\mu} = R^\alpha{}_{\mu\alpha\nu}$.
Cosmic Horizon in Robertson-Walker Universe:	$\delta(t) = a(t) \int_{t_0}^{\infty} \frac{cdt'}{a(t')}$.

SECTION A - Answer SIX parts of this section

1.1) Consider the two-dimensional metric space $ds^2 = dr^2 + r^2 d\theta^2$, where (r, θ) are polar coordinates, $r \in [0, \infty]$ and $\theta \in [0, 2\pi]$. Determine the Christoffel symbols using a method of your choice.

[7 marks]

1.2) What is the metric of space time in the *exterior* of a spherically symmetric non-rotating pulsating star? Justify your answer with a brief mathematical explanation in the context of General Relativity.

[7 marks]

1.3) A photon of frequency 10^{12} Hz is emitted at a given time by an observer located at a point on the surface of the Earth, and is received at a later time by an observer who lies directly above the observer on Earth at a height $h = 1274$ meters. Assume both observers to be static with respect to Earth as well as to each other, and ignore cosmological expansion. Also assume that the Earth is a spherically symmetric non rotating body. At what frequency will the second observer receive the photon? Explain the various steps of your analysis.

[7 marks]

1.4) In d space-time dimensions, the Einstein equations in the presence of matter without a cosmological constant read

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}.$$

in the standard notation, with G_N the Newton constant. *Conformal matter* is defined as the kind of matter for which the trace of the stress-tensor $\Theta \equiv g^{\mu\nu} T_{\mu\nu}$ *vanishes.* Show that in two space-time dimensions any matter satisfying the Einstein equations is *necessarily* conformal.

[7 marks]

1.5) Under general coordinate transformations $x^\mu \to x'^\mu(x^\nu)$, a scalar Φ and a covariant second rank tensor $\mathcal{T}_{\mu\nu}$ transform, by definition, as follows: $\Phi \to \Phi\,'(x') = \Phi(x)$ and $\mathcal{T}_{\mu\nu} \to \mathcal{T}\,'_{\mu\nu}(x') = (\partial x^\alpha / \partial x'^\mu)(\partial x^\beta / \partial x'^\nu) \mathcal{T}_{\alpha\beta}$ respectively. Consider the object $\Phi(x)\mathcal{T}_{\mu\nu}(x)$ and determine its transformation properties under general coordinate transformations. If $\mathcal{T}_{\mu\nu}$ is an antisymmetric second rank covariant tensor, what is the value of $g^{\mu\nu}\mathcal{T}_{\mu\nu}$, where $g^{\mu\nu}$ is the inverse of the metric tensor?

[7 marks]

1.6) Consider a four-dimensional *flat* ($k = 0$) Robertson-Walker space time $ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$ where χ is the radial coordinate. State and derive Hubble's law in such a Universe.

[7 marks]

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1.7) State the various forms of the Equivalence Principle of General Relativity. Using the appropriate form, determine the ratio of the mass in the presence of a gravitational field to that at spatial infinity in each of the following cases: (i) a ball of gold of radius r_0, and (ii) a neutron star of the same radius.

[7 marks]

1.8) Consider the two-dimensional metric: $ds^2 = \frac{1}{t^2}(dt^2 - dx^2)$. Geodesics are, by definition, curves $x(t)$ of extremal length, i.e. satisfying the variational equation:

$$\delta \int (ds^2)^{1/2} = \delta \int \frac{dt}{t} \sqrt{1 - \left(\frac{dx}{dt}\right)^2} = 0 \ .$$

Write down the Lagrange equation obtained from this variational principle, and thus show, without computing the Christoffel symbols, that the geodesics are given by :

$$(x - x_0)^2 = t^2 + a^2 \ ,$$

where $x_0, \ a$ are constants.

[*Hint*: solve the variational (Lagrange) equation by letting $\frac{dx}{dt} \equiv \tanh\theta$].

[7 marks]

SECTION B - Answer TWO questions

2) A two-dimensional space time is described by the infinitesimal line element:

$$ds^2 = -dt^2 + t dr^2,$$

where t is the time coordinate.

(a) Compute the Christoffel symbols for the above space time, by any method you prefer, and write down the appropriate geodesics.

[6 marks]

(b) Compute the independent components of the Riemann tensor for this two dimensional geometry.

[6 marks]

(c) For this space time show that the components of the Ricci tensor are:

$$R_{tt} = \frac{1}{4t^2}, \qquad R_{rr} = -\frac{1}{4t}, \qquad R_{tr} = R_{rt} = 0.$$

[6 marks]

(d) Compute the curvature scalar, $R = g^{\mu\nu} R_{\mu\nu}$, for this space time.

[6 marks]

(e) Describe the evolution of the universe governed by the above metric. Is there a cosmic horizon in this case?

[6 marks]

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3) An aircraft is flying back and forth for 15 hours (as measured by observers on the ground) at an altitude $h = 9000$ meters and with velocity $v = 140$ (meters)(sec)$^{-1}$. The atomic clocks carried by the plane are compared with identical clocks on the ground.

(a) Ignoring general relativistic effects, compute the special relativistic time dilation between the airborne and terrestrial clocks.

[5 marks]

(b) Assume that the plane flew very slowly, so that to a very good approximation it can be considered as being static at the altitude h above the Earth's surface. Treat the Earth as a spherical non-rotating body, of mass $M_\oplus$ and radius $r_\oplus$. Moreover, consider that the altitude h is very small as compared to the radius of the Earth, so that $h + r_\oplus \simeq r_\oplus$ to a good approximation.

Show that, as a consequence of General-Relativistic effects alone, during the t_{shell}=15 hour flight, the plane's clocks *gain* approximately

$$dt_{shell} \simeq \left(\frac{M_\oplus h}{r_\oplus^2}\right) t_{shell} \simeq 52.2 \times 10^{-9} \text{ sec},$$

as compared with the ground clocks.

[23 marks]

(c) Compare the special and general relativistic results in parts (a) and (b) above by expressing the special relativistic result as a percentage correction to the general relativistic one. Thus give the final result for the time of flight measured by the clock on the aircraft.

[2 marks]

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4) Assuming matter to behave as a perfect fluid, the Einstein equations for a four-dimensional perfect-fluid Friedmann–Robertson–Walker (FRW) cosmology are:

$$-3\frac{\dot{a}^2(t)}{a^2(t)} - 3\frac{k}{a^2(t)} + \Lambda = -8\pi G_N \rho \, , \tag{4.1}$$

$$-2\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} - \frac{k}{a^2(t)} + \Lambda = 8\pi G_N p \, , \tag{4.2}$$

Here the overdots denote derivatives with respect the cosmic time t, $a(t)$ is the scale factor, k is the usual characteristic parameter of the FRW cosmology, G_N is the Newton constant, Λ is the cosmological constant, ρ is the energy density, and p is the pressure.

(a) *Assume*, without proof, the following thermodynamic equation for producing work in this perfect fluid: $dE = -pdV$ where $V = a^3$ is the proper volume, and E is the total energy included in this proper volume. Using this equation determine the dependence of ρ on the scale factor $a(t)$ in the case of a perfect fluid universe with the equation of state: $p = w\rho$, where $w < 1$ is a time-independent positive constant.

[10 marks]

(b) For an *expanding* Universe, with zero cosmological constant $\Lambda = 0$, assume that it is *radiation dominated* ($w = 1/3$), and flat (k=0). By algebraically manipulating the system of equations (4.1),(4.2) show that in this case $\ddot{a}/a = -(\dot{a}/a)^2$, and thus determine the dependence of $a(t)$ and $\rho(t)$ on the cosmic time t.

[*Hint*: assume $a(t) = a_0 t^\ell$, a_0=const, ℓ=const, and determine ℓ].

[12 marks]

(c) What do you conclude about the acceleration and the cosmic horizon for this universe?

[4 marks]

(d) Consider now a *static* universe, with a generic value of k and in the presence of a positive cosmological constant $\Lambda > 0$. Show in this case that equations (4.1),(4.2) imply that *normal* matter with positive energy density can only exist if the universe is *closed*, that is: $k > 0$.

[4 marks]

5) Consider the motion of an orbiting planet, of mass m, in a Schwarzschild space time which describes the exterior of a massive, spherically symmetric celestial object of mass $M >> m$. Fix $\theta = \frac{\pi}{2}$ in the Schwarzschild metric and thus consider an effective three-dimensional Schwarzschild space time (t, r, ϕ), with $\phi = [0, 2\pi]$ the azimuth. Work in units where $G_N = c = 1$.

(a) Write down the Lagrangian for the planet, viewed as a satellite point particle in this geometry, and show that the conservation of energy and angular momentum are obtained as a consequence of Lagrange's equations for t and ϕ respectively.

[*Hint:* You may use the appropriate definitions given in the rubric]

[6 marks]

(b) (i) Use the conservation equations for energy and angular momentum (derived above) in the expression for the time-like invariant element $d\tau^2$ in the Schwarzschild geometry to arrive at the following equation:

$$(\frac{dr}{d\tau})^2 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\{1 + \left(\frac{L}{m}\right)^2 \frac{1}{r^2}\}$$

[3 marks]

(ii) From this equation determine the effective potential per unit satellite mass $\mathcal{U}/m$ (c.f. rubric) for the orbiting planet in terms of the energy and angular momentum of the planet.

[1 mark]

(c) Discuss the behaviour of the effective potential near the point $r \to 0$ and its physical significance for a satellite in the Schwarzschild geometry. From this explain which one of the two curves of figure 1 (see next page) corresponds to a sketch of the Schwarzschild space-time effective potential under consideration.

[6 marks]

(d) Consider the case in which the energy of the satellite is slightly above the local minimum, but well below the local maximum of the effective potential. Provide a *rough sketch* of the orbit in this case.

[1.5 marks]

QUESTION CONTINUES ON NEXT PAGE

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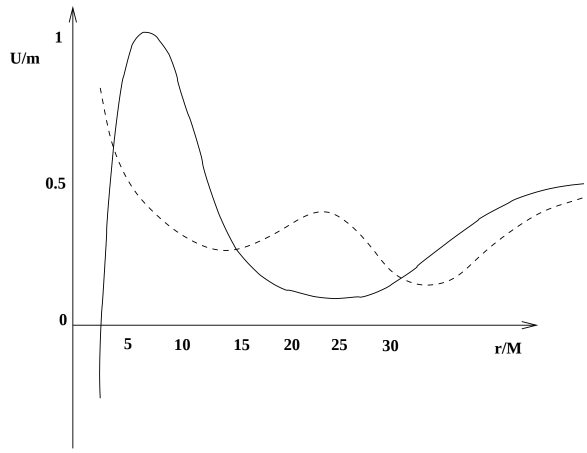

Figure 1: Two possible curves (dashed or continuous) purporting to be the Schwarzschild effective potential.

(e) Approximate the radial motion near the local minimum of the effective potential at $r = r_0$ by a *harmonic oscillation motion* of frequency $\omega_r^2 \simeq d^2(\mathcal{U}/m)/dr^2|_{r\to r_0}$. Assume, *without proof*, that the radial (ω_r) and angular (ω_ϕ) oscillation frequencies are given by:

$$\omega_r^2 = \frac{M(r_0 - 6M)}{r_0^3(r_0 - 3M)} , \qquad \omega_\phi^2 = \frac{M}{r_0^2(r_0 - 3M)} .$$

(i) Compute the difference $\Delta\omega = \omega_\phi - \omega_r$ of these two rates, by making the approximation: $\omega_\phi^2 - \omega_r^2 \simeq 2\omega_\phi(\omega_\phi - \omega_r)$ and thus show that

$$\Delta\omega \simeq \frac{3M}{r_0}\omega_\phi .$$ [3.5 marks]

(ii) Use this to determine the precession angle in terms of the total angle covered in orbital motion. [3 marks]

(f) Consider the planet Mercury as it orbits the Sun. Determine in *degrees* the advance precession of the perihelion of Mercury per century (=100 Earth years). The (average) radius of Mercury's orbit is $r_0 = 5.8 \times 10^{10}$ meters, the period of Mercury's orbit around the Sun is 7.6×10^6 sec, and the period of Earth around the Sun is 3.156×10^7 sec.

[6 marks]

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