King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3212 Statistical Mechanics

Summer 2006

Time allowed: THREE Hours

**Candidates should answer ALL parts of SECTION A,
and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.**

The approximate mark for each part of a question is indicated in square brackets.

**You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.**

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ϵ_0	=	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	=	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_e	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_u	=	1.661×10^{-27}	kg
Proton rest mass	m_p	=	1.673×10^{-27}	kg
Neutron rest mass	m_n	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_B	=	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_A	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

Throughout $\beta = \frac{1}{k_B T}$ and T is the temperature.

SECTION A – Answer ALL parts of this section

1.1) For N very weakly interacting fermions of mass m in a volume V the Fermi energy μ is given by

$$\mu = \left(\frac{N}{V}\right)^{\frac{2}{3}} \frac{h^2}{2m} \left(\frac{3}{8\pi}\right)^{\frac{2}{3}} .$$

Calculate in electron volts the Fermi energy of aluminium given that the mass density is 2700 kg m^{-3}, its relative atomic mass is 27 and there are three conduction electrons per aluminium atom.

[5 marks]

1.2) For a one-dimensional harmonic oscillator with frequency ω show that the partition function Z is given by

$$Z = \frac{\exp\left(-\frac{\beta\hbar\omega}{2}\right)}{1 - \exp\left(-\beta\hbar\omega\right)} .$$

[5 marks]

1.3) A paramagnetic solid in a magnetic field of strength B contains N weakly interacting particles, each with a permanent magnetic moment $m\sigma$; σ can have the $2S+1$ values $-S, -S+1, \ldots, S-1, S$. Show that the partition function of the system can be written as

$$Z = \left(\frac{\sinh\left(S + \frac{1}{2}\right) Bm\beta}{\sinh\left(\frac{1}{2} Bm\beta\right)}\right)^N .$$

[10 marks]

1.4) A system consists of two distinguishable atoms. Each atom can exist in three quantum energy eigenstates, a ground state with energy taken to be 0 and a doubly degenerate excited state with energy ε. Determine the state space of the system and the partition function.

[10 marks]

1.5) For an ideal gas of N classical monatomic particles of mass m in three dimensions, calculate the density of states $g(E)$ where E is the total energy of the system.

[5 marks]

1.6) A three-dimensional harmonic oscillator has energy levels

$$\varepsilon_{n_1,n_2,n_3} = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right)$$

where each n_i can be $0, 1, 2, \ldots$. Find the degeneracies of the levels of energy $7\hbar\omega/2$ and $9\hbar\omega/2$. Given that the system is in thermal equilibrium at temperature T show that the higher energy level is more populated than the lower one if $\ln(5/3)\, k_B T > \hbar\omega$.

[5 marks]

SECTION B – Answer TWO questions

2) For a very weakly interacting gas of fermions the condition for degeneracy is

$$n \gg \left(\frac{2\pi m k_B T}{h^2}\right)^{\frac{3}{2}}$$

where n is the number density, m is the mass of each fermion and T is the temperature.

A hypothetical white dwarf star is supposed to consist of $^{29}\mathrm{Si}_{14}$ at a temperature of 10^9 K and a mass density of 10^{10} kg m^{-3}. The material is completely ionised. Use the condition for degeneracy to determine whether the perfect gas equation of state is appropriate for the electron gas and the nuclear gas.

[20 marks]

Given that for a degenerate gas the pressure p is $\frac{2}{5} n E_F$ where E_F is the Fermi energy, calculate the contribution to the internal pressure of the star due to the electron gas and the nuclear gas.

[10 marks]

3) A system of weakly interacting indistinguishable particles obeys Bose-Einstein, Fermi-Dirac or Maxwell-Boltzmann statistics. For such systems write down the definitions of the partition function in a grand canonical ensemble with temperature T and chemical potential μ.

[3 marks]

Evaluate the partition function for each statistic.

[6 marks]

Argue that the probability distribution $P_i\left(n_i, T, \mu\right)$ for finding n_i particles in a given single-particle state labelled by i with energy ε_i, is given by

a)

$$P_i\left(n_i, T, \mu\right) = \frac{\exp\left(-\beta\left[\varepsilon_i - \mu\right] n_i\right)}{\left(1 + a\exp\left(-\beta\left[\varepsilon_i - \mu\right]\right)\right)^a}$$

with $a = 1$ for fermions and $a = -1$ for bosons;

and

b)

$$P_i\left(n_i, T, \mu\right) = \frac{\exp\left(-\beta\left[\varepsilon_i - \mu\right] n_i\right)}{\exp\left(e^{-\beta\left[\varepsilon_i - \mu\right]}\right) n_i!}$$

for Maxwell-Boltzmann particles.

[3 marks]

For each case, use the distribution to find expressions for the average occupation number $\langle n_i \rangle$.

[6 marks]

Express P_i in each case as a function of n_i and $\langle n_i \rangle$.

[3 marks]

Obtain an expression for the relative fluctuation in the occupation number $\Delta n_i / \langle n_i \rangle$ where $\Delta n_i = \sqrt{\left\langle \left(n_i - \langle n_i \rangle\right)^2 \right\rangle}$.

[9 marks]

4) In a cavity of macroscopic size show that the grand canonical partition function for photons is given by

$$Z = \prod_i \left(1 - e^{-\beta \varepsilon_i}\right)^{-1}$$

where ε_i is the energy of the ith single particle state, $\beta = \frac{1}{k_B T}$ and T is the temperature.

[7 marks]

In a 3-dimensional cubic cavity of volume V show that the single particle density of states $g(\varepsilon)$ in energy ε is

$$g(\varepsilon) = aV\varepsilon^2$$

where $a = \frac{1}{\pi^2 \hbar^3 c^3}$.

[7 marks]

Find expressions for the pressure P, energy density u, entropy density s and specific heat C_V per unit volume of black-body radiation at temperature T on using

$$\begin{aligned} P &= \frac{k_B T}{V} \log Z \\ u &= -\frac{1}{V} \frac{\partial \log Z}{\partial \beta} \\ s &= \frac{1}{V} \left[\frac{\partial (PV)}{\partial T} \right]_V \\ C_V &= \left(\frac{\partial u}{\partial T} \right)_V \end{aligned}$$

and

$$\int_0^\infty dx\, x^2 \log\left(1 - e^{-x}\right) = -\frac{\pi^4}{45}.$$

[16 marks]

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