King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3212 Statistical Mechanics

Summer 2005

Time allowed: THREE Hours

Candidates should answer all SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ϵ_0	=	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	=	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_{e}	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_{u}	=	1.661×10^{-27}	kg
Proton rest mass	m_{p}	=	1.673×10^{-27}	kg
Neutron rest mass	m_{n}	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_{B}	=	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

Free energy F in terms of the N-particle partition function Z_N:

$$F = -k_{\mathrm{B}} T \log Z_N$$

The internal energy U in terms of the N-particle partition function Z_N:

$$U = k_{\mathrm{B}} T^2 \frac{\partial}{\partial T} \log Z_N$$

The entropy S in terms of the N-particle partition function Z_N:

$$S = k_{\mathrm{B}} \log Z_N + k_{\mathrm{B}} T \frac{\partial}{\partial T} \log Z_N$$

The number of photons $n(\omega)\,d\omega$ in a frequency interval $(\omega, \omega + d\omega)$ is given by

$$n(\omega)\,d\omega = \frac{V}{\pi^2 c^3} \omega^2 \frac{d\omega}{e^{\frac{\hbar\omega}{k_{\mathrm{B}} T}} - 1}$$

Stirling's formula:

$$\ln n! \sim n \ln n - n \quad \text{as } n \to \infty$$

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SECTION A – Answer all SIX parts of this section

1.1) Show that the partiton function Z for a single non-relativistic particle of mass m in an infinite one-dimensional square-well potential of width L is

$$Z = \sum_{n=1}^{\infty} e^{-\frac{n^2\theta}{T}}$$

where $\theta = \frac{\hbar^2\pi^2}{2mL^2k_B}$, T is the temperature and quantum mechanics can be presumed to apply.

Hint: the energy levels of the particle in the potential are given by $E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}$.

[7 marks]

1.2) Consider an ideal system of a thousand non-interacting spin $\frac{1}{2}$ particles in the absence of an external magnetic field. Derive an expression for the probability that exactly n of the thousand particles has spin up.

[7 marks]

1.3) The energy levels of the five spin states associated with a particle of spin two have energies $2\varepsilon, \varepsilon$, 0, $-\varepsilon$ and -2ε. Find an expression for the free energy at temperature T of a system consisting of N such particles, each at a different site, assuming that they are non-interacting.

[7 marks]

1.4) Two identical non-interacting spin $\frac{1}{2}$ fermion particles with spin up, can occupy any of three single particle states. Two of these states have energy ε and the other has energy 0. Calculate the partition function when the system is at temperature T.

[7 marks]

1.5) When the Universe expands by a linear factor χ, all wavelengths are stretched by the factor χ.

Show that the mean energy density is mutiplied by a factor χ^{-4}.

Hence show that the black body radiation filling the Universe remains that of a black body but with a temperature which scales as χ^{-1}.

[7 marks]

1.6) By differentiating the single particle partition function $Z_{sp} = \sum_i e^{-\beta\varepsilon_i}$ with respect to $\beta \left(= \frac{1}{k_B T}\right)$ show that for a system of N particles obeying Boltzmann statistics, the internal energy U is given by

$$U = N k_{\mathrm{B}} T^2 \frac{\partial}{\partial T} \log Z_{sp}.$$

[7 marks]

SECTION B – Answer TWO questions

2a) A system consists of N identical but distinguishable particles each of which has two energy levels. The energy separation of the levels is fixed at ε and the upper level is g-fold degenerate.

Calculate the number of states with total energy $E = n\varepsilon$ where the energy of the lower level is taken to be 0.

[9 marks]

b) By using Stirling's formula when N and n are large show that the entropy $S(E,N)$ is given by

$$S(E,N) = Nk_B\left(x\ln x + (1-x)\ln(1-x) - x\ln g\right)$$

where $x = {}^{n}\!/_{N}$.

[10 marks]

The temperature T associated with the system can be identified through the relation

$$\frac{1}{k_\mathrm{B}T} = \frac{1}{k_\mathrm{B}}\left(\frac{\partial S}{\partial E}\right)_N .$$

c) Solve for x in terms of T and hence find the occupation number of the lower energy level in terms of T.

[5 marks]

d) If $g = 2$ and $E = 0.75N\varepsilon$ find an expression for T proving that it is negative.

[2 marks]

f) If the system is brought into contact with a bath at equilibrium at any temperature discuss the direction of the flow of heat between it and the bath.

[4 marks]

3) Consider a linear chain of $N+1$ atoms each of mass m for which the atoms at each end of the chain are fixed. Let ξ_i be the displacement of the i-th atom from its equilibrium position. The potential energy V of the chain is given by

$$V = \frac{1}{2} K \sum_{i=0}^{N-1} \left(\xi_{i+1} - \xi_i\right)^2$$

where K is a positive constant.

a) Show that the equation of the i-th atom is

$$m\ddot{\xi}_i + 2K\xi_i - K\left(\xi_{i+1} + \xi_{i-1}\right) = 0.$$

[5 marks]

b) By making the transformation

$$\xi_l = \sqrt{\frac{2}{N}} \sum_{j=1}^{N-1} x_j \sin\left(\frac{lj\pi}{N}\right)$$

prove that

$$\ddot{x}_j = -\omega_j^2 x_j, \quad j = 1, \ldots, N-1$$

where $\omega_j = \sqrt{\frac{4K}{m}} \sin\left(\frac{j\pi}{2N}\right)$.

[5 marks]

c) In terms of these variables the system can be considered to be a set of $N-1$ independent harmonic oscillators. Assuming the form of the energy eigenvalues for a single harmonic oscillator in quantum theory, show that, for this system, the energy of a given configurtion $\{n_j\}$ is

$$E\left(\{n_j\}\right) = \sum_{j=1}^{N-1} \left(n_j + \frac{1}{2}\right)\hbar\omega_j.$$

[5 marks]

d) Prove that the canonical partition function Z is given by

$$Z = \prod_{j=1}^{N-1} \frac{e^{-\frac{1}{2}\beta\hbar\omega_j}}{1 - e^{-\beta\hbar\omega_j}}$$

where $\beta = \frac{1}{k_{\mathrm{B}}T}$ and T is the temperature.

[10 marks]

e) Show that the internal energy $U(T)$ can be written as

$$U(T) = \int_0^{\omega_{\max}} d\omega \; g(\omega) \left(\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\right)$$

where $\omega_{\max} = \sqrt{\frac{4\kappa}{m}}$ and $g(\omega)$ is a suitable density of states.

[5 marks]

4a) A white dwarf star consists mainly of ^4He which is completely ionised. The pressure resisting the inward pull of gravity is due to the electrons.

Consider such a star with mass density $\rho = 10^{10}$ Kg m^{-3} and temperature $T = 10^7$ K. Using the density of states in k-space and transforming into the corresponding one in energy show that the number density n of non-relativistic electrons is given by

$$n = \frac{1}{2\pi^2}\left(\frac{2m_e}{\hbar^2}\right)^{3/2}\int_0^\infty d\varepsilon \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}+1}$$

where m_e is the mass of the electron and μ is the chemical potential.

[8 marks]

b) Hence show that, at $T = 0$, the Fermi level is determined by the relation

$$n = \frac{1}{3\pi^2}\left(\frac{2m_e E_F}{\hbar^2}\right)^{3/2}.$$

[8 marks]

c) Demonstrate (using numerical values) the degeneracy condition

$$k_{\mathrm{B}}T \ll E_F$$

i.e. T is negligible.

[3 marks]

d) The pressure p of the electron gas is given by

$$p = \frac{2k_BT}{(2\pi)^3}\int d^3k \ln\left[1+\exp\left(-\frac{\left(\varepsilon\left(\overleftarrow{k}\right)-\mu\right)}{k_BT}\right)\right]$$

where $\varepsilon\left(\overleftarrow{k}\right)$ is the energy of a single electron with momentum $\overleftarrow{k}$. For the case of the degenerate electron gas show that

$$p \approx \frac{1}{5}\left(3\pi^2\right)^{\frac{2}{3}}\frac{\hbar^2}{m_e}n^{\frac{5}{3}}$$

[8 marks]

and equivalently that

$$p \approx \kappa\rho^{\frac{5}{3}}$$

[3 marks]

with $\kappa = \frac{1}{5}\left(3\pi^2\right)^{\frac{2}{3}}\frac{\hbar^2}{m_e}\left(\frac{1}{2m_p}\right)^{\frac{5}{3}}$ where m_p is the proton mass.