King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3201 Mathematical Methods in Physics III

Summer 2006

Time allowed: THREE Hours

Candidates should answer ALL parts of SECTION A,
and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ϵ_0	=	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	=	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_{e}	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_{u}	=	1.661×10^{-27}	kg
Proton rest mass	m_{p}	=	1.673×10^{-27}	kg
Neutron rest mass	m_{n}	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_{B}	=	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

SECTION A – Answer ALL parts of this section

1.1) Find the residue at $z = 1$ of the function $1/(z-1)(z-2)$.

[4 marks]

1.2) Define the Lagrangian of a system in general and, for the particular case of a harmonic oscillator in 3-dimensions, state its form using cartesian co-ordinates.

[6 marks]

1.3) From the Laurent series

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n$$

deduce that

$$J_{n-1}(z) - J_{n+1}(z) = 2\frac{d}{dz}J_n(z)\,.$$

[7 marks]

1.4) Use the Cauchy theorem to calculate

$$\oint_C e^{\frac{1}{z}}\, dz$$

where C is the contour $|z-3| = 2$.

[5 marks]

1.5) Use the calculus of variations to show that the shortest path between two points in a plane is a straight line.

[7 marks]

1.6) Show that the function $x - iy$ does not satisfy the Cauchy-Riemann relations.

[4 marks]

1.7) Evaluate the contour integral

$$\oint_{|z|=3} dz \, \frac{1}{(z-1)\left(\frac{z}{2}-1\right)}.$$

[7 marks]

SECTION B – Answer TWO questions

2) A circular membrane of radius 2 lies in a region of the xy plane with plane polar coordinates $0 \le \rho \le 2$ and $0 \le \varphi \le 2\pi$. The boundary of the membrane is fixed. The membrane may be assumed to vibrate according to the equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \varphi^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

where ψ is the vertical displacement of the membrane and c is a constant. Show that the normal modes of vibration of the membrane are

$$\psi_{m,s}(\rho, \varphi, t) = A_{m,s} J_m(k_{m,s}\rho) \sin(m\varphi + B_{m,s}) \cos(ck_{m,s}t + D_{m,s})$$

where $m = 0, 1, 2, \ldots$, $s = 0, 1, 2, \ldots$, $A_{m,s}$, $B_{m,s}$ and $D_{m,s}$ are constants, $k_{m,s} = j_{m,s}/2$ and $\{j_{m,s}; s = 1, 2, \ldots\}$ are the positive zeros of the Bessel function $J_m(z)$.

Hint: the differential equation

$$z^2 f'' + zf' + \left(z^2 - m^2\right) f = 0$$

has a solution $f(z) = AJ_m(z) + BY_m(z)$ where A and B are constants.

[20 marks]

Show that the radial part of the normal mode $\psi_{m,s}(\rho, \varphi, t)$ satisfies the orthogonality relation

$$\int_0^2 J_m\left(j_{m,r}\frac{\rho}{2}\right) J_m\left(j_{m,s}\frac{\rho}{2}\right) \rho \, d\rho = 0,$$

where $r, s = 1, 2, \ldots$ and $r \ne s$.

[10 marks]

3) State the Cauchy residue theorem for contour integrals.

[3 marks]

Use this theorem to evaluate the following integrals

a)

$$\int_0^{2\pi} \frac{\exp(-2i\theta)}{(5 - 3\sin\theta)^2}\, d\theta,$$

[12 marks]

b)

$$\int_0^{\infty} \frac{x^2}{1 + x^4}\, dx.$$

[15 marks]

Justify the neglect of any contour integral in part (b).

4a) A classical system with n degrees of freedom has a lagrangian $L(\{q_i\}, \{\dot{q}_i\})$ with $1 \leq i \leq n$; define the corresponding hamiltonian and deduce the Hamilton equations of motion.

[5 marks]

b) A particle of mass m is constrained to move on the surface of a smooth torus which has a parametric representation

$$x = \rho \cos\psi, \quad y = \rho \sin\psi, \quad z = \sin\theta$$

where

$$\rho = 2 + \cos\theta$$

with $0 \leq \psi < 2\pi$ and $0 \leq \theta < 2\pi$. Apart from the constraint there are no other external forces acting on the particle. In terms of generalised co-ordinates θ and ψ show that the lagrangian of the particle is

$$L = \frac{m}{2}\left[(2 + \cos\theta)^2 \dot{\psi}^2 + \dot{\theta}^2\right]$$

[12 marks]

c) Determine the hamiltonian of the system.

[7 marks]

d) From the Hamilton equations show that if the particle moves with $\theta = 0$ then $\dot{\psi}$ is a constant of motion.

[6 marks]