King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP3201 Mathematical Methods in Physics III

Summer 2003

Time allowed: THREE Hours

Candidates must answer SIX parts of SECTION A,
and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Determine all of the values of the complex logarithm $\ln\left(1 - e^{\frac{i\pi}{2}}\right)$.

[7 marks]

1.2) Locate and classify all of the singularities in the finite z-plane of the function

$$f(z) = \frac{\left(z^2 + 3z + 2\right)\sin^2 z}{z^2\left(z^2 - 1\right)^2}.$$

[7 marks]

1.3) Determine a Laurent series for the function

$$f(z) = \frac{1}{(z+3)(z+5)}$$

which is valid in the region $0 < |z+3| < 2$.

[7 marks]

1.4) Derive the expression for the integral

$$\int_0^\infty \exp\left(-ax^4\right) dx$$

(where a is a positive real number) in terms of the gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

[7 marks]

[Hint: Make the substitution $t = x^4$.]

1.5) Use the method of Lagrange multipliers to find the area of the largest rectangle that can be inscribed within the ellipse

$$\frac{x^2}{100} + \frac{y^2}{64} = 1.$$

[7 marks]

1.6) State the Cauchy-Riemann equations for an analytic function. Use these equations to determine whether

$$\frac{y + ix}{x^2 + y^2}$$

is an analytic function.

[7 marks]

1.7) Use the Bessel function series

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} \left(\frac{z}{2}\right)^{2m+n},$$

where $\Gamma(x)$ denotes the gamma function, to derive the relation

$$\frac{d}{dz}\left[z^{-1} J_1(z)\right] = -z^{-1} J_2(z).$$

[7 marks]

1.8) Three masses m_1, m_2, and m_3 lie on a straight line and are connected by two springs of stiffness K. The potential energy V for a spring has the form

$$V = \frac{1}{2} K (x_1 - x_2)^2 + \frac{1}{2} K (x_2 - x_3)^2$$

where x_1, x_2, and x_3 are the displacements of the respective masses from their equilibrium positions. Write down the lagrangian for the system in terms of these displacements. From the Lagrange equations show that

$$m_2 \ddot{x}_2 + K(x_2 - x_1) - K(x_3 - x_2) = 0.$$

[7 marks]

SECTION B – Answer TWO questions

2) State the Cauchy residue theorem.

[5 marks]

Consider a function $f(z)$ which is analytic at the real integral values of $z = 0, \pm1, \pm2, \ldots$ and tends to zero at least as fast as $|z|^{-2}$ as $|z| \to \infty$.

Show that

$$\oint_S \pi\cot(\pi z)\, f(z)\, dz = 2\pi i \left\{ \sum_{n=-N}^{+N} f(n) + \begin{matrix} \big(\text{sum of the residues of } \pi\cot(\pi z)\, f(z) \\ \text{at the poles of } f(z) \text{ inside } S\big) \end{matrix} \right\}$$

where S is a square with corners at the points $z = \left(N + \frac{1}{2}\right)(\pm1 \pm i)$ where N is any positive integer.

[15 marks]

[Hint: Use the Cauchy residue theorem. Note that there are two types of poles, one arising as poles of $\cot(\pi z)$ and the other as poles of $f(z)$.]

On assuming that $\oint_S \pi\cot(\pi z)\, f(z)\, dz \to 0$ as $N \to \infty$ and by considering $f(z) = \frac{1}{z^2+1}$ show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1 + \pi\coth\pi}{2}.$$

[10 marks]

3) The Bessel function $J_p(ax)$ satisfies the differential equation

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\left(a^2x^2-p^2\right)y=0$$

and also assume the recursion relation

$$\frac{d}{dx}J_p(x)=\frac{p}{x}J_p(x)-J_{p+1}(x).$$

Use this to prove the following orthogonality theorem:

$$\int_0^1 xJ_p(ax)\;J_p(bx)\;dx=\begin{cases}0\text{ if }a\neq b\\ \frac{1}{2}J_{p+1}^2(a)\text{ if }a=b\end{cases}$$

where a and b are zeros of $J_p(x)$.

[20 marks]

Hence evaluate

$$\int_0^1\left(\frac{\sin ax}{ax}-\cos ax\right)^2dx,$$

where a is a root of the equation $\tan x = x$.

[10 marks]

[Hint: Note that $J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$.]

4) Define the hamiltonian in terms of the lagrangian. Show that the hamiltonian is independent of velocities.

[5 marks]

A particle of mass m is constrained to move on a smooth surface which has a parametric representation

$$x = \rho\cos 2\phi,\ y = \rho\sin 2\phi,\ z = k\rho$$

where $0 < \rho < \infty,\ 0 \leq \phi \leq \pi$ and k is a positive constant. The force of gravity acts in the negative z direction.

a) Derive an expression for the lagrangian of the system. Hence obtain the Lagrange equations of motion.

[15 marks]

b) Deduce that

$$m\rho^2 \frac{d}{dt}\phi = \frac{J}{2}$$

where J is a constant of motion and hence

$$m\left(1+k^2\right)\ddot{\rho} = \frac{J^2}{m\rho^3} - mgk$$

[5 marks]

c) Write $u = \frac{1}{\rho}$ and hence show that a particle trajectory $\rho = \rho(\phi)$ satisfies the differential equation

$$\frac{\left(1+k^2\right)}{4}\frac{d^2u}{d\phi^2} = -u + \frac{m^2gk}{J^2u^2}.$$

[5 marks]

[Hint: In terms of generalised cordinates the Lagrange equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0.]$$

5) The real and imaginary parts of an analytic function $f(z)$ satisfy Laplace's equation. Use this to outline a method of solving Laplace's equation in an open connected region of the plane.

[10 marks]

A slab of material of constant thermal conductivity is located in the region

$$-\tfrac{1}{2} \le x \le \tfrac{1}{2}$$
$$0 \le y < \infty.$$

The temperature Φ satisfies Laplace's equation within the material and at the boundaries of the slab is given by

$$\Phi = \begin{cases} T, & x = -\frac{1}{2} \\ 2T, & x = \frac{1}{2} \\ 0, & y = 0 \end{cases}$$

Assume that the mapping

$$w = \sin(\pi z)$$

transforms the region of the slab (when $z = x + iy$ is considered as the complex plane) into the upper half plane of the w-plane.

By solving the Laplace equation in the w-plane and on using the mapping, prove that in the steady state

$$\Phi = \frac{T}{\pi}\tan^{-1}\left\{\frac{\cos\pi x\,\sinh\pi y}{\sin\pi x\,\cosh\pi y\,+1}\right\} - \frac{2T}{\pi}\tan^{-1}\left\{\frac{\cos\pi x\,\sinh\pi y}{\sin\pi x\,\cosh\pi y\,-1}\right\} + 2T.$$

[20 marks]

[Hint: In the w-plane consider the imaginary parts of $\ln(w+1)$ and $\ln(w-1)$ in order to satisfy Laplace's equation.]