# King's College London

UNIVERSITY OF LONDON

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**B.Sc. EXAMINATION**

**CP/2260 Mathematical Methods in Physics II**

**Summer 2003**

**Time allowed: THREE Hours**

**Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.**

**The approximate mark for each part of a question is indicated in square brackets.**

**You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.**

**TURN OVER WHEN INSTRUCTED**
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## SECTION A – Answer SIX parts of this section

1.1) A general curvilinear orthogonal coordinate system $(q_1, q_2, q_3)$ is given by the transformation functions:

$$x = x(q_1, q_2, q_3), \;\; y = y(q_1, q_2, q_3), \;\; z = z(q_1, q_2, q_3).$$

Show that the unit base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and the scale factors $h_1$, $h_2$, $h_3$ in this system are given, respectively, by the following relations:

$$\mathbf{e}_i = \frac{1}{h_i}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right), \;\; h_i = \left|\frac{\partial \mathbf{r}}{\partial q_i}\right|.$$

[7 marks]

1.2) Consider the spherical polar coordinates $(r, \theta, \phi)$,

$$x = r\sin\theta\cos\phi, \;\; y = r\sin\theta\sin\phi, \;\; z = r\cos\theta,$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Determine the scale factors $h_r$, $h_\theta$, $h_\phi$ for this system and the unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$.

[7 marks]

1.3) Evaluate the integral

$$\int_{-\infty}^{\infty} \delta\left(\frac{x}{5} - 1\right) f(x)\mathrm{d}x$$

where $f(x)$ is any continuous function.

[7 marks]

1.4) Give a formal definition of the integral Fourier transform, $F(\nu) = \mathcal{F}\left[f(t)\right]$, and its inverse, $f(t) = \mathcal{F}^{-1}\left[F(\nu)\right]$, for a function $f(t)$ defined in the interval $-\infty < t < \infty$. Hence derive the following integral representation for the delta function:

$$\delta(t) = \int_{-\infty}^{\infty} e^{\mathrm{i}2\pi\nu t}\mathrm{d}\nu$$

[7 marks]

1.5) Show that the Fourier transform of the function $f(t) = e^{-\alpha|t|}$ is

$$\mathcal{F}\left[f(t)\right] = \frac{2\alpha}{\alpha^2 + (2\pi\nu)^2}$$

Hint: when calculating the transform, split the integral over $t$ into two: one with $t \leq 0$ and another with $t \geq 0$.

[7 marks]

1.6) Specify and classify the singular points of the differential equation

$$(x^2 - 1)(x - 4)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (x - 1)\frac{\mathrm{d}y}{\mathrm{d}x} + (x + 1)y = 0$$

[7 marks]

1.7) Verify that the function

$$y(x, t) = y_1(x + ct) + y_2(x - ct)$$

with arbitrary functions $y_1(x)$ and $y_2(x)$ is a solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

[7 marks]

1.8) Use the generating function

$$G(x, t) = \frac{1}{\sqrt{1 - 2xt + t^2}}$$

for the Legendre polynomials $P_n(x)$ to derive explicit expressions for the polynomials with $n = 0, 1$.

[7 marks]

## SECTION B – Answer TWO questions

2) The *parabolic coordinates* $(u, v, \theta)$ are specified by the following transformation equations:

$$x = uv\cos\theta, \quad y = uv\sin\theta, \quad z = \frac{1}{2}\left(u^2 - v^2\right),$$

where $u \geq 0$, $v \geq 0$ and $0 \leq \theta < 2\pi$ .

a) Show that the scale factors $h_u$, $h_v$, $h_\theta$ have the form:

$$h_u = h_v = \sqrt{u^2 + v^2}, \quad h_\theta = uv$$

[9 marks]

b) Show that the unit base vectors $\mathbf{e}_u$, $\mathbf{e}_v$, $\mathbf{e}_\theta$ can be expressed via the Cartesian vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as follows:

$$\mathbf{e}_u = \frac{v}{\sqrt{u^2 + v^2}}\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j} + \frac{u}{v}\mathbf{k}\right)$$

$$\mathbf{e}_v = \frac{u}{\sqrt{u^2 + v^2}}\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j} - \frac{v}{u}\mathbf{k}\right)$$

$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

[9 marks]

c) Show that the parabolic coordinate system is orthogonal.

[4 marks]

d) Given that the Laplacian operator $\nabla^2$ of a scalar field $\Psi(x, y, z)$ in the general othogonal curvilinear coordinates $(q_1, q_2, q_3)$ is

$$\nabla^2\Psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Psi}{\partial q_3}\right)\right]$$

show that the Laplace equation $\nabla^2\Psi(x, y, z) = 0$ can be rewritten in the parabolic coordinates in the following form:

$$\frac{1}{u^2 + v^2}\left[v\frac{\partial}{\partial u}\left(u\frac{\partial\Psi}{\partial u}\right) + u\frac{\partial}{\partial v}\left(v\frac{\partial\Psi}{\partial v}\right)\right] + \frac{1}{uv}\frac{\partial^2\Psi}{\partial\theta^2} = 0$$

[8 marks]

3) Let $F(\nu) = \mathcal{F}[f(t)]$ be the Fourier transform of a function $f(t)$. You may assume that $f(t)$, its first and second order derivatives tend to zero at $\pm\infty$.

a) Prove the following identity which is sometimes called the *modulation theorem*:

$$\mathcal{F}[f(t)\cos(2\pi\nu_0 t)] = \frac{1}{2}[F(\nu+\nu_0) + F(\nu-\nu_0)].$$

[6 marks]

b) Express the Fourier transforms of $f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t}$ and $f''(t) = \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}$ in terms of $F(\nu)$. Hint: when calculating the Fourier transforms, use integration by parts.

[10 marks]

c) The $n$-th moment of a function $f(t)$ is given by the expression:

$$\mu_n = \int_{-\infty}^{\infty} t^n f(t)\mathrm{d}t$$

By considering derivatives of the Fourier transform $F(\nu)$, show that

$$\mu_n = \frac{F^{(n)}(0)}{(-2\pi \mathrm{i})^n},$$

where $F^{(n)}(0)$ is the $n$-th derivative of $F(\nu)$ calculated at $\nu = 0$.

[8 marks]

d) Calculate the Fourier transform of the function $f(t)$ which is equal to unity if $0 \le t \le 1$ and zero otherwise.

[6 marks]

4) Consider the *Bessel differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

a) Classify the singular points of this equation.

[4 marks]

b) Assuming that the parameter $p$ in the Bessel equation is either noninteger or a negative number, use the Frobenius method to derive three first terms of **two** independent series solutions of the equation, $y_1(x)$ and $y_2(x)$.

[24 marks]

c) Hence, state the general solution of the equation.

[2 marks]

5) Consider the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

for a string of length $L$ fixed at both ends.

a) Use the Fourier method of separation of variables to obtain two ordinary differential equations: one involving $t$ and another $x$.

[6 marks]

b) Given that the separation constant $k = -p_n^2$, where $p_n = \frac{\pi}{L}n$ and $n = 1, 2, 3, ...$, check that the appropriate solution of the equation involving $x$ which is consistent with the boundary conditions is $\psi_n(x) = \sin(p_n x)$.

[4 marks]

c) Check that

$$\chi_n(t) = A_n \sin(cp_n t) + B_n \cos(cp_n t)$$

is a solution of the equation involving $t$, where $A_n$ and $B_n$ are arbitrary constants.

[3 marks]

d) Now assume that initially (i.e. at $t = 0$) the string is pulled by 0.06 units at $x = L/5$ and then released, i.e. the initial conditions are:

$$y(x, 0) = 0.3\frac{x}{L}$$

for $0 \le x \le L/5$ and

$$y(x, 0) = 0.075\left(1 - \frac{x}{L}\right)$$

for $L/5 \le x \le L$ and also $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$. Given that the general solution of the wave equation is

$$y(x, t) = \sum_{n=1}^{\infty} \psi_n(x)\chi_n(t),$$

determine the corresponding partial solution of the wave equation.
Hint: the integral $\int x \sin(p_n x)\, \mathrm{d}x$ is calculated by parts.

[17 marks]

You may assume that functions $\psi_n(x)$ satisfy the orthogonality relation:

$$\int_0^L \psi_n(x)\psi_{n'}(x)\mathrm{d}x = \frac{L}{2}\delta_{nn'}$$