King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP2250 Mathematical Methods in Physics I

Summer 2004

Time allowed: 3 Hours

Candidates should answer no more than SIX parts of SECTION A, and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ϵ_0	$=$	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	$=$	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	$=$	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	$=$	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	$=$	1.602×10^{-19}	C
Electron rest mass	m_{e}	$=$	9.109×10^{-31}	kg
Unified atomic mass unit	m_{u}	$=$	1.661×10^{-27}	kg
Proton rest mass	m_{p}	$=$	1.673×10^{-27}	kg
Neutron rest mass	m_{n}	$=$	1.675×10^{-27}	kg
Planck constant	h	$=$	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_{B}	$=$	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	$=$	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	R	$=$	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	$=$	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		$=$	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	$=$	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

Definition of Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}\left[a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)\right]$$

where

$$a_0 = \frac{2}{T}\int_c^{c+T} f(x)dx$$

$$a_n = \frac{2}{T}\int_c^{c+T} f(x)\cos\left(\frac{2\pi n x}{T}\right)\mathrm{d}x$$

$$b_n = \frac{2}{T}\int_c^{c+T} f(x)\sin\left(\frac{2\pi n x}{T}\right)\mathrm{d}x.$$

Stokes′s Theorem $\displaystyle\int_C \mathbf{A}.d\mathbf{r} = \int_S \nabla \times \mathbf{A}.d\mathbf{S}.$

Gauss′s Theorem $\displaystyle\int_V \nabla.\mathbf{A}\, d\mathbf{r} = \int_S \mathbf{A}.d\mathbf{S}.$

SECTION A – Answer SIX parts of this section

1.1) Determine whether the matrix

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

is

i) hermitian,

ii) unitary.

[7 marks]

1.2) Solve the differential equation

$$\frac{dy}{dx} = \frac{y+1}{x-1}$$

given the boundary condition $y = 2$ at $x = 0$.

[7 marks]

1.3) Evaluate the line integral

$$I = \int_C (2xy\,dy - x^2\,dx)$$

over the contour made up of the three sides of the triangle with vertices at $O(0,0)$, $P(2,0)$ and $Q(2,2)$.

[7 marks]

1.4) Evaluate the surface integral

$$\int\!\!\int_S \nabla \times \mathbf{F}.d\mathbf{S}$$

where $\mathbf{F} = 3y\,\mathbf{i} - 2x\,\mathbf{j} + xy\,\mathbf{k}$ and S is the hemispherical surface described by $x^2 + y^2 + z^2 = 9$ with $z \geq 0$. [Hint: Use Stokes's theorem to consider a bounding surface at $z = 0$.]

[7 marks]

1.5) Use Gauss's theorem, or otherwise, to evaluate $\int_S \mathbf{A}.d\mathbf{S}$ for the vector field $\mathbf{A} = 2xy\,\mathbf{i} - y^2\,\mathbf{j} + (z + xy)\,\mathbf{k}$ for a cylindrical region of base radius 3 and of height extending from $z = 0$ to $z = 5$.

[7 marks]

1.6) Solve the differential equation

$$\frac{dy}{dx} + \frac{5}{x}y = x^2$$

given that $y = 1/8$ when $x = 1$.

[7 marks]

1.7) Find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and show that $\mathbf{x_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{x_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are eigenvectors.

[7 marks]

1.8) For the range $-T/2 \leq t \leq T/2$ the Fourier series representation $F\left[f(t)\right]$ for the function $f(t) = \cos\left(\pi t/T\right)$ is given by:

$$F\left[f(t)\right] = \frac{2}{\pi} + \frac{1}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^2 - 1/4}\cos\left(\frac{2n\pi t}{T}\right)$$

Verify that the average value of this waveform is $2/\pi$ and comment on how the other terms in the Fourier series might resemble the time-varying output from a full-wave rectifier electronic circuit.

[7 marks]

SECTION B – Answer TWO questions

2) Three identical springs of equilibrium lengths l and force constants s are attached end-to-end between two rigid supports $3l$ apart. Two identical particles each of mass m, are attached at the ends of the middle spring and are constrained to move along the direction of the springs. If the displacements of each of the particles from its equilibrium position are x_1 and x_2 respectively, show that the equations of motion of the two particles are given by:

$$m\frac{d^2x_1}{dt^2} = s(x_2 - 2x_1)$$

$$m\frac{d^2x_2}{dt^2} = s(x_1 - 2x_2).$$

[7 marks]

Show how the above equations can be written into the matrix form:

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{\Lambda x}$$

and write down the matrix $\mathbf{\Lambda}$.

[5 marks]

By considering the eigenvalues of $\mathbf{\Lambda}$ and by assuming harmonic solutions of the form $\mathbf{x(t)} = \mathbf{p}\exp(i\omega t)$, determine the characteristic frequencies of the system.

[8 marks]

Evaluate the eigenvectors of the system and comment on the physical significance of their forms.

[10 marks]

3) Given the following differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 8x\exp 3x$$

i) determine the complementary function solution, and

[10 marks]

ii) determine the particular integral solution.

[20 marks]

4) State the Cayley-Hamilton theorem for square matrices and verify it for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}.$$

[8 marks]

Show that the third power of a matrix, which has a non-zero determinant and a non-zero trace, can be formed from linear combinations of that matrix and the identity matrix. Use this result to evaluate $\mathbf{A^3}$.

[12 marks]

Use the Cayley-Hamilton theorem to express the inverse of the matrix $\mathbf{A}$ in terms of linear combinations of $\mathbf{A}$ and the identity matrix $\mathbf{I}$ and hence write down $\mathbf{A^{-1}}$.

[10 marks]

5) The periodic function $f(x)$, of period 2π, is defined in the region $-\pi < x < \pi$ by $f(x) = \exp(x)$. Outside this region, $f(x)$ is defined by the requirement of periodicity. Show that the even function coefficients, a_0 and a_n, of the Fourier series for $f(x)$ are given by:

$$a_0 = \frac{2}{\pi} \sinh(\pi),$$

[7 marks]

$$a_n = \frac{2}{\pi} \frac{(-1)^n}{(1+n^2)} \sinh(\pi).$$

[9 marks]

State the Dirichlet convergence in the mean. Show that this implies that the value of this series at the discontinuity points $x = \pi$ and $x = -\pi$ is $\cosh(\pi)$.

[5 marks]

Given that the odd function coefficients are given by:

$$b_n = \frac{2n}{\pi} \frac{(-1)^{n+1}}{(1+n^2)} \sinh(\pi)$$

show by evaluating $\cosh(\pi)$ by means of the Fourier series that

$$\coth(\pi) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(1+n^2)}.$$

[9 marks]