King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP2250 Mathematical Methods in Physics I

Summer 2003

Time allowed: 3 Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Definition of Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}\left[a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)\right]$$

where

$$a_0 = \frac{2}{T}\int_c^{c+T} f(x)dx$$

$$a_n = \frac{2}{T}\int_c^{c+T} f(x)\cos\left(\frac{2\pi n x}{T}\right)\mathrm{d}x$$

$$b_n = \frac{2}{T}\int_c^{c+T} f(x)\sin\left(\frac{2\pi n x}{T}\right)\mathrm{d}x$$

Paragraph spacing

SECTION A – Answer SIX parts of this section

1.1) Solve the following differential equation for $y(x)$ subject to the boundary condition $y(0) = 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\exp(y)}{x^2+1}.$$

[7 marks]

1.2) Show that the eigenvalues of any (2×2) Hermitian matrix $\mathbf{A}$ are distinct except when $\mathbf{A}$ is a multiple of the identity matrix $\mathbf{I}$.

[7 marks]

1.3) Showing your working, determine which of the following sets of vectors are linearly independent:

i) $\mathbf{a} = 2\mathbf{j} + 2\mathbf{k},\ \mathbf{b} = \mathbf{i} + \mathbf{k},\ \mathbf{c} = \mathbf{i} + \mathbf{j}$.

ii) $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k},\ \mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k},\ \mathbf{c} = 7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$.

[7 marks]

1.4) Show that the component of $\nabla\phi(x,y)$ in the $\mathbf{i}$ direction of the surface $\phi(x,y) = \exp -(x^2+y^2)$ is:

$$-\sqrt{2}/e$$

at the point $\mathbf{r} = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$.

[7 marks]

1.5) Find the complementary function solution to the following differential equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = (1+x)\exp(x).$$

[7 marks]

1.6) Determine which of the following fields is conservative:

a) $\mathbf{H} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

b) $\mathbf{E} = -y\mathbf{i} + x\mathbf{j}$.

[7 marks]

1.7) Evaluate the path integral $\int \mathbf{E}.\mathrm{d}\mathbf{r}$ along the path $y = x$ from the point $(0,0)$ to $(2,2)$ for the field $\mathbf{E}$ in question 1.6 (b).

[7 marks]

1.8) Show that in the range $0 \leq x < \pi$ the Fourier series representation for the function $f(x)$ defined as:

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi/2 \\ 0 & x = \pi/2 \\ -1 & \pi/2 < x \leq \pi \end{cases}$$

is given by:

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos(nx).$$

[7 marks]

SECTION B – Answer TWO questions

2) State the *Cayley-Hamilton* theorem for square matrices.

[5 marks]

Verify the Cayley-Hamilton theorem for the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}.$$

[10 marks]

The matrix $\mathbf{A}$ can be diagonalised by a similarity transform. Show that the eigenvalues of $\mathbf{A}$ are $\lambda = 3$ and $\lambda = -1$ and that the similarity matrix $\mathbf{T}$ is given by:

$$\mathbf{T} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}.$$

Hence use the similarity transform to evaluate $\mathbf{A}^{1/2}$.

[15 marks]

3) A radioactive element decays into a second element, which then subsequently decays into a third stable element. The numbers of atoms of each element are respectively N_1, N_2 and N_3 with corresponding decay rates k_1 and k_2 for the first two elements. The amount of each element is given by the following set of coupled differential equations:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -k_1 N_1,$$

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = +k_1 N_1 - k_2 N_2,$$

$$\frac{\mathrm{d}N_3}{\mathrm{d}t} = +k_2 N_2.$$

Show that these equations can be transformed into a matrix operator equation of the form:

$$\mathbf{Ba} = \mathbf{\Lambda a}.$$

where $\mathbf{a}$ is a vector.

Explicitly write out the matrix operator $\mathbf{B}$.

[5 marks]

Use an eigenvalue and eigenvector analysis to find the solution for the numbers of each type of element as a function of time.

[15 marks]

[Hint: You may assume that the behaviour of N as a function of time is of the form $N = \mathbf{a}\exp(\lambda t)$, where $\mathbf{a}$ and λ are to be determined.]

To find any constants use the initial condition that, at $t = 0, N_1 = N_0, N_2 = N_3 = 0$.

[10 marks]