King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

B.Sc. EXAMINATION

CP1400 Classical Mechanics and Special Relativity

Summer 2005

Time allowed: THREE Hours

Candidates should answer ALL parts of SECTION A,
and TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ϵ_0	$=$	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	$=$	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	$=$	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	$=$	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	$=$	1.602×10^{-19}	C
Electron rest mass	m_{e}	$=$	9.109×10^{-31}	kg
Unified atomic mass unit	m_{u}	$=$	1.661×10^{-27}	kg
Proton rest mass	m_{p}	$=$	1.673×10^{-27}	kg
Neutron rest mass	m_{n}	$=$	1.675×10^{-27}	kg
Planck constant	h	$=$	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_{B}	$=$	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	$=$	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	R	$=$	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	$=$	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		$=$	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	$=$	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

Throughout this examination paper, t denotes time and dots over a letter denote derivatives with respect to time.

SECTION A – Answer SIX parts of this section

1.1) Assuming the invariance of Newton's second law under Galilean transformations

$$t' = t$$

$$\vec{r}\,' = \vec{r} + \vec{V}t + \vec{r}_0,$$

show that the force is independent of the inertial observer.

[7 marks]

1.2) Write down an expression for the power applied by a force $\vec{f}$ to a particle of mass m moving with velocity $\vec{v}$. Show that the power is equal to the time derivative of the kinetic energy of the particle.

[7 marks]

1.3) Derive an expression for the angular velocity of a satellite orbiting the Earth on a circular trajectory of radius R. Show that the period of the orbit is

$$T = 2\pi \frac{R^{3/2}}{\sqrt{GM}},$$

where M is the mass of the Earth.

[7 marks]

1.4) In complex notation, the equation of motion of a forced oscillator is

$$\ddot{x} + \frac{\dot{x}}{\tau} + \omega^2 x = g \exp(i\Omega t),$$

where τ, ω, g and Ω are constants. Assuming a solution of the form

$$x = A \exp i(\Omega t + \phi),$$

derive an expression for A in terms of g, ω, Ω and τ. State the physical conditions that apply when the forced oscillator is in a state of resonance.

[7 marks]

1.5) A solid is rotating around a fixed point O, under the influence of a force $\vec{f}$ applied at another point A of the solid. Define the torque of the force $\vec{f}$ with respect to O. Assuming the solid to be made up of a collection of point particles, show that the torque is equal to the time derivative of the angular momentum of the solid about O.

[7 marks]

1.6) The Lorentz transformations between any two inertial frames (t, x, y, z) and (t', x', y', z') are

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z,$$

with the usual notation. Show that

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - v}{1 - \frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}},$$

and confirm that the speed of light is the same in all inertial frames.

[7 marks]

SECTION B – Answer TWO questions

2) The Earth, of radius R and mass M, rotates with angular velocity ω in an inertial frame. A projectile of mass m is launched from a point on the equator, radially outwards in the rotating frame of the Earth. The only force that acts on the projectile is the gravitational attraction of the Earth, and eventually the projectile falls back on Earth to a different point also on the equator.

a) Show that the angular momentum is conserved and explain why the motion is planar.

[6 marks]

b) Let (r, θ) be the polar coordinates of the projectile in the inertial frame where the Earth rotates, with the centre of the Earth as the origin of the coordinates. From the conservation of angular momentum, show that $r^2\dot{\theta} = R^2\omega$.

[6 marks]

c) Show that the total energy of the projectile is $E = \frac{1}{2}m(\dot{r})^2 + U_{\text{eff}}(r)$, where

$$U_{\text{eff}}(r) = \frac{1}{2}m\frac{R^4\omega^2}{r^2} - \frac{GmM}{r}.$$

[6 marks]

d) From the conservation of energy, show that the maximum distance d that the projectile reaches from the centre of the Earth before falling back is

$$d = \frac{1}{2E}\left[-GmM + \sqrt{(GmM)^2 + 2mR^4\omega^2 E}\right]$$

[8 marks]

e) Sketch the general form of the effective potential $U_{\text{eff}}(r)$ and explain how to find the distance d graphically.

[4 marks]

3) A homogeneous wheel of mass m and radius R can roll along a horizontal rail which is turning around a vertical axis with angular velocity Ω (see figure). The centre of the wheel is connected to the axis of rotation by a spring of constant k. The motion of the wheel is studied in the rotating frame of the rail and the position of the wheel's centre of mass is at a distance x from the axis of rotation.

a) Show that the moment of inertia of the wheel with respect to a perpendicular axis through its centre is $I = \frac{1}{2}mR^2$.

[5 marks]

b) If the wheel moves without sliding, show that the kinetic energy of the wheel is $E_{\text{kin}} = \frac{3}{4}m(\dot{x})^2$.

[6 marks]

c) The natural length (at rest) of the spring is l. Show that the potential energy corresponding to the spring force is $U_1 = \frac{1}{2}k(x-l)^2$.

[4 marks]

d) The inertial force that has to be taken into account due to the rotation of the rail is $-ma$, where a is the centripetal acceleration of the point on the rail which coincides with the contact point of the wheel. Show that this inertial force may be derived from the potential $U_2 = -\frac{1}{2}m\Omega^2x^2$.

[5 marks]

e) By differentiating the total energy with respect to time, derive an differential equation for $x(t)$ and show that, for $m\Omega^2 < k$, the wheel oscillates with angular frequency

$$\omega = \sqrt{\frac{2}{3}\left(\frac{k}{m} - \Omega^2\right)}.$$

[6 marks]

f) For $m\Omega^2 < k$, give an expression for the equilibrium position of the wheel.

[4 marks]

4) Two clocks at rest in an inertial frame $\mathcal{S}$ are separated by the proper length l_0, in the x direction. To synchronize the clocks in $\mathcal{S}$, a fixed source of light placed halfway between them, emits a pulse of light in both directions, at $t = 0$ measured in $\mathcal{S}$. An observer moving along the x direction with constant speed v, coincides with the location of the source of light at the time the source emits the pulses. The observer measures the distance l between the clocks.

a) Give an expression, in terms of l_0, for the time t_0 measured in $\mathcal{S}$, at which both clocks receive the ray of light.

[4 marks]

b) Show that in the rest frame of the moving observer the time for the ray of light to reach the clock that the observer is moving towards is

$$t_1 = \frac{1}{2}\frac{l}{c+v}.$$

[5 marks]

c) Show that in the rest frame of the moving observer the time for the ray of light to reach the clock that the observer is moving away from is

$$t_2 = \frac{1}{2}\frac{l}{c-v}.$$

[5 marks]

d) State the relationship between l_0 and l in terms of v, and derive an expression for the difference Δt between the times the two clocks receive the pluses, as measured by the moving observer.

[4 marks]

e) Sketch a Minkowski diagram, showing both inertial frames and specifying the angle between the axes.

[4 marks]

f) Show graphically, on a new Minkowski diagram, how the two clocks synchronized in $\mathcal{S}$ fail to be synchronized for the moving observer.

[4 marks]

g) Show graphically, on a third Minkowski diagram, why $l \neq l_0$.

[4 marks]