King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP1400 Classical Mechanics and Special Relativity

Summer 2004

Time allowed: THREE Hours

Candidates should answer no more than SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ϵ_0	=	8.854×10^{-12}	$\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	=	$4\pi \times 10^{-7}$	$\mathrm{H\,m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\,m^2\,kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_e	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_u	=	1.661×10^{-27}	kg
Proton rest mass	m_p	=	1.673×10^{-27}	kg
Neutron rest mass	m_n	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k_B	=	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_A	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^3}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\,m^{-2}}$

Throughout this examination paper, t denotes time and dots over a letter denote derivatives with respect to time.

SECTION A – Answer SIX parts of this section

1.1) A car is moving on a road, which is an inertial frame, with an acceleration $\vec{a}$. The driver, of mass m, is subject to his/her weight $m\vec{g}$ and the reaction $\vec{R}$ of the car. Write down a vector equation of motion for the driver in the road's frame. Explain the occurrence of inertial forces that the driver experiences in the car's frame.

[7 marks]

1.2) A general solution of harmonic oscillations is $x(t) = x_0 \cos(\omega t + \phi_0)$, where x_0 and ϕ_0 are constants of integration and ω is the angular frequency. Derive an expression for the specific solution corresponding to the initial conditions $x(0) = 0$ and $\dot{x}(0) = v$.

[7 marks]

1.3) A point particle of mass m experiences a force $\vec{f}$ and has a velocity $\vec{v}$. The power of the force is give by $\mathcal{P} = \vec{f} \cdot \vec{v}$. Show that $\mathcal{P}$ is the rate of change with time of the kinetic energy of the particle.

[7 marks]

1.4) Consider a solid body as a collection of point particles rotating about a common axis. Show that its kinetic energy in the centre of mass frame is $E_k = \frac{1}{2}I\omega^2$, where I is the moment of inertia with respect to the axis of rotation of the body and ω is its angular velocity.

[7 marks]

1.5) A vertical pulley wheel of radius R can rotate around its axis of symmetry with moment of inertia I. A rope of negligible mass is wound around the pulley and has a mass m tied at the other end. The rope does not slip on the pulley wheel and the total energy of the system is

$$E = \frac{1}{2}I\omega^2 + \frac{1}{2}m(\dot{z})^2 + mgz,$$

where ω is the angular velocity of the pulley wheel and z is the vertical coordinate of the mass. Considering the conservation of energy, derive an expression for the angular acceleration $\dot{\omega}$ of the pulley.

[7 marks]

1.6) The equation for electric and mechanical harmonic oscillations are analogous, replacing the coordinate x by the electric charge q, the mass m by the impedance L and the spring force constant k by the inverse capacitance C^{-1}. Using these analogies and the equation of motion of a frictionless mechanical harmonic oscillator, give an equation describing the electric oscillations. Hence derive an expression for their angular frequency as a function of L and C.

[7 marks]

1.7) A point particle A of mass m and velocity $\vec{v}$ moves in an inertial frame with origin O under the influence of a central force $\vec{f}$. Show that the angular momentum of the particle is conserved and explain why the motion is planar.

[7 marks]

1.8) The total energy of a relativistic free point particle of mass m and speed v is $E = \gamma mc^2$, where c is the speed of light and $\gamma = [1 - (v/c)^2]^{-1/2}$. Using the small x expansion $(1+x)^n \simeq 1+nx+n(n-1)x^2/2$, give the three contributions to the energy and their physical interpretations.

[7 marks]

SECTION B – Answer TWO questions

2) Consider different possible trajectories of a comet of mass m, moving around the Sun of mass M. Let $\mathcal{L}$ be the angular momentum of the comet and $r(t)$ its distance from the Sun. The motion can be described by the energy conservation equation $E = \frac{1}{2}m(\dot{r})^2 + U(r)$, where E is the total energy of the comet and

$$U(r) = -\frac{GMm}{r} + \frac{\mathcal{L}^2}{2mr^2}$$

is the effective potential.

a) Explain qualitatively why the total energy E and the angular momentum $\mathcal{L}$ are conserved.

[4 marks]

b) Plot the effective potential and show that its minimum occurs at a distance

$$R = \frac{\mathcal{L}^2}{GMm^2}.$$

Hence derive an expression for the energy E_0 of a comet on a circular trajectory of radius R.

[8 marks]

c) A comet has energy E_1 such that $E_0 < E_1 < 0$. Explain, using a new plot of the effective potential, why the position of this comet oscillates between two distances R_1 and R_2. To what trajectory does this situation correspond?

[6 marks]

d) Derive expressions for R_1 and R_2, as functions of E_1 and $\mathcal{L}$.

[6 marks]

e) A particular comet has zero total energy. Show that the minimum distance between the comet and the sun is $R_0 = R/2$. To what trajectory does this situation correspond?

[6 marks]

3) Pedestrians marching over the Millennium Bridge cause a vertical oscillation of frequency Ω, such that the displacement z of the middle of the bridge satisfies the equation

$$\ddot{z} + \frac{\dot{z}}{\tau} + \omega^2 z = a\cos(\Omega t),$$

where ω, τ are constants characterizing the bridge, and a is a constant characterizing the pedestrians' march.

a) From the equation of motion, determine the dimensionality of the constants ω, τ, a.

[3 marks]

b) Explain qualitatively why steady-state solutions of the equation of motion can be approximated by $z \simeq A\cos(\Omega t + \phi)$.

[5 marks]

c) By writing the equation of motion, and the steady-state solution in complex notations, show that

$$A = \frac{a\tau^2}{\sqrt{\tau^4(\Omega^2 - \omega^2)^2 + (\Omega\tau)^2}}.$$

[8 marks]

d) Hence show that the amplitude V of the velocity $\dot{z}$ of the bridge is

$$V = \frac{a\tau}{\sqrt{1 + (\omega\tau)^2\left(\frac{\omega}{\Omega} - \frac{\Omega}{\omega}\right)^2}}.$$

[6 marks]

e) Define resonance. Under what condition does resonance occur in the amplitude V?

[4 marks]

e) Explain why, if $\omega\tau >> 1$, the resonance generated by the pedestrians is very sharp. Plot V as a function of Ω/ω for two different values of τ, indicating which curve corresponds to the larger value of τ.

[4 marks]

4) A cylindrical body of mass M and radius R has a moment of inertia I with respect to its axis of symmetry. It rolls down a hill of height h under the acceleration g due to gravity. The motion occurs without slipping.

a) The speed of the centre of mass of the body is v and its angular velocity is ω. Explain why $v = R\omega$.

[5 marks]

b) Write down an expression for the total energy E of the body when its height is z. Explain why E is conserved.

[5 marks]

c) The body starts at the top of the hill with zero velocity. Using conservation of energy, show that when it arrives at the bottom of the hill the speed of its centre of mass is

$$v = \sqrt{\frac{2gh}{1 + \frac{I}{MR^2}}}.$$

[5 marks]

d) Derive expressions for the moment of inertia I and the speed at the bottom of the hill for

(i) a hollow cylinder, with walls of negligible thickness,

[4 marks]

(ii) a solid homogeneous cylinder.

[8 marks]

e) Explain why any arbitrary mass distribution would still give a speed at the bottom independent of the mass and the radius of the cylinder.

[3 marks]

5) Twins wish to check Einstein's theory of Special Relativity (SR). One is on Earth, which is supposed to be an inertial frame, and the other is orbiting in a rocket around the Earth, on a circular trajectory, with constant speed v. During each infinitesimal time interval $\mathrm{d}t$ measured by the twin on Earth, it may be assumed that the other twin is in an inertial frame following a motion tangential to the circular trajectory, with a constant velocity $\vec{v}$ with respect to the Earth, such that $|\vec{v}| = v$.

a) Define the proper time τ for the orbiting twin.

[3 marks]

b) A point particle displaced a distance $\mathrm{d}\vec{r}$ during time $\mathrm{d}t$ from the point of view of the twin on Earth, is displaced by $\mathrm{d}\vec{\rho}$ during time $\mathrm{d}\tau$ from the point of view of the orbiting twin. SR requires that $-(c\mathrm{d}\tau)^2 + (\mathrm{d}\vec{\rho})^2 = -(c\mathrm{d}t)^2 + (\mathrm{d}\vec{r})^2$. Show that the proper time τ of the orbiting twin satisfies the equation

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}}.$$

[6 marks]

c) By considering the different tangential inertial frames at each moment τ, show that

$$\Delta\tau = \Delta t\sqrt{1 - \frac{v^2}{c^2}},$$

where Δt is the finite time interval measured by the twin on Earth, corresponding to the finite time interval $\Delta\tau$ measured by the orbiting twin.

[5 marks]

d) The speed of the rocket with respect to the Earth is $v = 3 \times 10^3$ m s^{-1} and the trip of the orbiting twin lasts 10 years, measured in the Earth's frame. Estimate, in seconds, the difference in the twins' ages after this trip.

[6 marks]

e) What should the velocity of the orbiting twin be, to give an age difference of one year compared with the twin on Earth?

[5 marks]

f) The so-called twin paradox states that if this experiment is described from the point of view of the twin in the rocket, it should be the twin on Earth whose age is less, when they meet after 10 years. Discuss this paradox.

[5 marks]