Atomic & Molecular Physics Solutions to Problem Sheet 2224.3
Issued Thursday 21 February 2008, due Thursday 28 February 2008

1. The X-ray spectrum consists of a continuous background with sharp peaks on top (the characteristic spectrum). The background arises from bremmstrahlung - radiation emitted as the electrons are decelerated in the Coulomb field of the nuclei. The maximum energy of the bremmstrahlung radiation will depend on the energy of the incident electrons. **[2]**

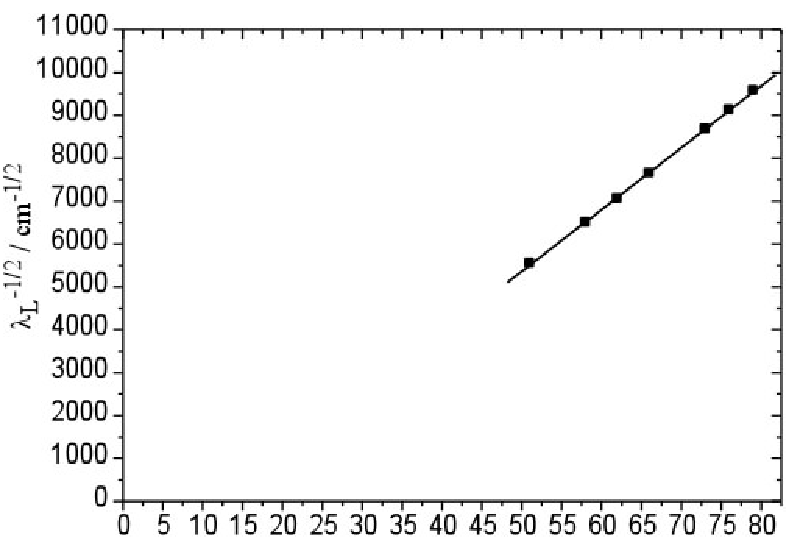
   The peaks occur when an inner shell electron is ejected by a collision, leaving the atom in a highly excited state. A higher shell electron will make a transition to the inner shell vacancy emitting an X-ray in the process. The frequency of this X-ray will depend on the energy difference of the initial and final states, and thus is a property of the target material. **[2]**

   Moseley's Law for the wavelength of the characteristic lines is:

$$\frac{1}{\lambda} = \tilde{R}(Z-S)^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

   where n_f and n_i are the principal quantum numbers of the final and initial quantum states.

   Moseley's method is to plot $\lambda^{-\frac{1}{2}}$ vs Z, which should be a straight line with non-zero intercept as shown below: **[4]**



   From Moseley's Law we can see that the gradient of this graph should be:

$$g = \sqrt{\tilde{R}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)}$$

   and from the fit to the graph we find that $g = 144.2$ cm$^{-1/2}$.

Knowing that $\tilde{R} = 109767$ cm^{-1} and that for the L-series lines $n_f = 2$ we find that:

$$\frac{1}{n_i^2} = \frac{1}{2^2} - \frac{144.2^2}{109767}$$

giving us that $n_i = 4$, and therefore these lines are L$_\beta$ lines ($\Delta n = 2$) **[1]**

The intercept on the horizontal axis is non-zero and will be equal to:

$$Z_0 = -\sqrt{\tilde{R}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)}S$$

which from the parameters of the fit we find to be $Z_0 = -1847.4$, giving a screening constant of

$$S = \left(\frac{1847.4}{144.2}\right) = 12.8$$

[1]

2. The Landé g-factors g_j is:

$$g_j = 1 + \left(\frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}\right)$$

So for $^2s_{\frac{1}{2}}$, $j = \frac{1}{2}, s = \frac{1}{2}, l = 0$, giving

$$g_j(^2s_{\frac{1}{2}}) = 1 + \left(\frac{\frac{1}{2}(\frac{1}{2}+1) + \frac{1}{2}(\frac{1}{2}+1) - 0(0+1)}{2 \times \frac{1}{2}(\frac{1}{2}+1)}\right) = 2$$

For $^2p_{\frac{1}{2}}$, $j = \frac{1}{2}, s = \frac{1}{2}, l = 1$, giving

$$g_j(^2s_{\frac{1}{2}}) = 1 + \left(\frac{\frac{1}{2}(\frac{1}{2}+1) + \frac{1}{2}(\frac{1}{2}+1) - 1(1+1)}{2 \times \frac{1}{2}(\frac{1}{2}+1)}\right) = \frac{2}{3}$$

For $^2p_{\frac{3}{2}}$, $j = \frac{3}{2}, s = \frac{1}{2}, l = 1$, giving

$$g_j(^2s_{\frac{1}{2}}) = 1 + \left(\frac{\frac{3}{2}(\frac{3}{2}+1) + \frac{1}{2}(\frac{1}{2}+1) - 1(1+1)}{2 \times \frac{3}{2}(\frac{3}{2}+1)}\right) = \frac{4}{3}$$

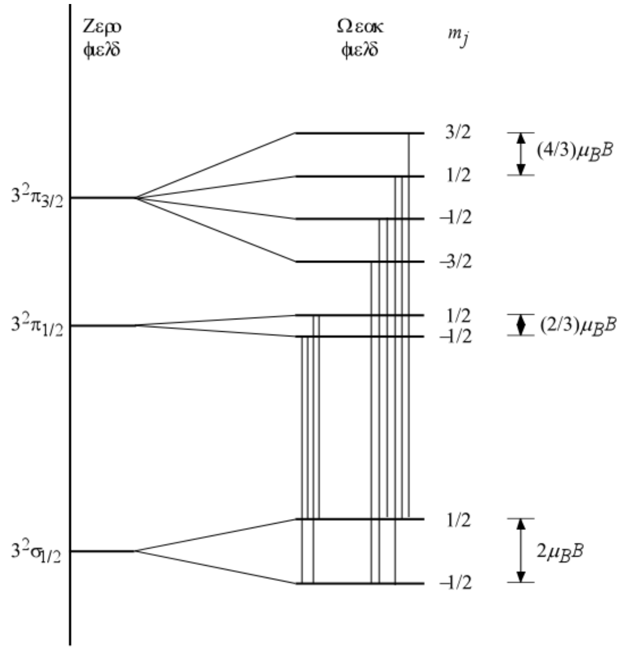
[3]

For the magnetic field to be 'weak', we require that it be less than the spin-orbit splitting of the excited state, thus

$$\mu_B B < \Delta E_{\text{LS}}, \text{ or } B < \frac{\Delta E_{\text{LS}}}{\mu_B}$$

We have that $\Delta E_{\text{LS}} = 17.2$ cm^{-1} $= 3.42 \times 10^{-22}$ J, giving:

$$B < \frac{3.42 \times 10^{-22}}{9.274 \times 10^{-24}}$$



i.e. $\underline{B < 37\text{ T}}$ **[2]**

The splitting of the energy levels and the allowed transitions are shown in the diagram below.

For the right diagram and allowed transitions: **[5]**

3. In a two-level atom, rate of absorption $= CU(\nu_{12})N_1$, rate of spontaneous emission $= AN_2$ and rate of stimulated emission $= BU(\nu_{12})N_2$, where $U(\nu_{12})$ is the spectral enegy density at the transition frequency.

   The rate of change of the population leaving the ground state is therefore:

$$-CU(\nu_{12})N_1$$

   and the rate of change of population leaving the excited state is:

$$-\left(AN_2 + BU(\nu_{12})N_2\right)$$

   so the net rate of change of population of the excited state (say) is:

$$\frac{dN_2}{dt} = \underbrace{CU(\nu_{12})N_1}_{\text{pop.increase}} - \underbrace{(AN_2 + BU(\nu_{12})N_2)}_{\text{pop.decrease}}$$

   and similarly for the ground state.

   In the *steady state*, $\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0$, and so:

$$CU(\nu_{12})N_1 = AN_2 + BU(\nu_{12})N_2,$$

and so we obtain for the ration of populations of the states:

$$\frac{N_1}{N_2} = \frac{AN_2 + BU(\nu_{12})N_2}{CU(\nu_{12})N_1}.$$

In *thermal equilibrium*, we know that:

$$\frac{N_1}{N_2} = \frac{\exp(-E_1/k_BT)}{\exp(-E_2/k_BT} = \exp(h\nu_{12}/k_BT)$$

so we may equate these giving:

$$A + BU(\nu_{12}) = CU(\nu_{12})\exp(h\nu_{12}/k_BT)$$

from which we obtain an expression for the spectral energy density:

$$U(\nu_{12}) = \frac{(A/B)}{(C/B)\exp(h\nu_{12}/k_BT) - 1}.$$

We can compare the above with the energy density of a black body (Planck's formula)

$$U(\nu_{12}) = \frac{8\pi\nu_{12}^2}{c^3}\frac{h\nu_{12}}{\exp(h\nu_{12}/k_BT) - 1}$$

and so identify the relations between the coefficients A, B and C:

$$\mathbf{C = B}$$

and

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{8\pi h\nu_{12}^3}}{\mathbf{c^3}}$$

[6]

The energy difference between the ground and excited states is

$$\Delta E = (20.66 - 18.70)\text{eV} = 1.96\text{eV} = 3.14 \times 10^{-19}\text{J}$$

The corresponding wavelength is:

$$\lambda = \frac{hc}{\Delta E} = \frac{6.626 \times 10^{-34} \times 2.998 \times 10^8}{3.14 \times 10^{-19}} = 632.6\text{nm}$$

[2]

The energy per photon is ΔE as calculated above, and the power is $P = 2.5 \times 10^{-3}\text{Js}^{-1}$. The rate of photons emitted is therefore:

$$r = \frac{P}{\Delta E} = \frac{2.5 \times 10^{-3}}{3.14 \times 10^{-19}} = 8 \times 10^{15}\text{s}^{-1}$$

[2]