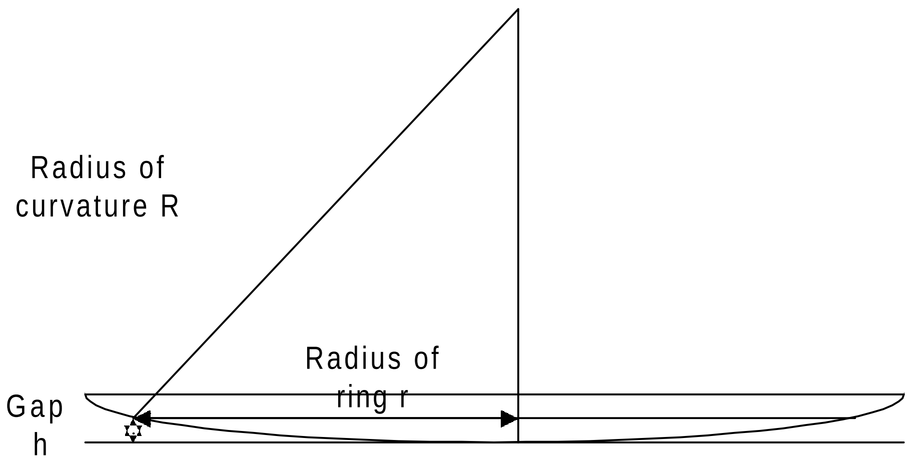
Model Solutions - 1B24 Waves, Optics and Acoustics

1. The partial differential equation describing wave motion in one dimension is
$c^2\, \partial^2 y/\partial x^2 = \partial^2 y/\partial t^2$. Substituting the given expression for y_1 1 mark
the left-hand side gives $-c^2 \cdot A \cdot k^2 \cdot sin(k \cdot x - \omega \cdot t)$
the right-hand side gives $-A \cdot \omega^2 \cdot sin(k \cdot x - \omega \cdot t)$ 1 mark
which satisfies the wave equation provided that the wave speed $c = \frac{\omega}{k}$. 1 mark

The difference between the two waves is that whereas y_1 is propagating to the right, y_2 is propagating to the left. The sum of the two is
$A \cdot sin(k \cdot x + \omega \cdot t) + A \cdot sin(k \cdot x - \omega \cdot t) = 2 \cdot A \cdot sin(k \cdot x) \cdot sin(\omega \cdot t)$
which is a standing wave, with wavelength $\lambda = 2\pi / k$ and angular frequency ω.
(1 mark for direction; 1 mark for manipulation; 2 marks for correct description of standing wave)

2. Reflection from the air-glass interface is reflection from an optically more dense medium, and involves a phase change of π. Reflection from the glass-air interface involves no phase change. 2 marks
Newton's rings may be described using the diagram below.



Elementary geometry tells us that $(2 \cdot R - h) \cdot h = r^2$, so that if we ignore the second order term in the small quantity h/R we have $r = \sqrt{2 \cdot R \cdot h}$.
The interference occurs between rays reflected from the lower surface of the lens (no phase change) and those which are reflected from the flat (phase change of π). Thus destructive interference (dark rings) occurs when the geometrical path difference $2h$ is an integral number of wavelengths $2n\lambda$.
That is
$r = \sqrt{n \cdot R \cdot \lambda}$. 3 marks for derivation; 2 marks for diagram or verbal explanation.

3. The acoustic impedance Z of a material of density ρ in which the speed of sound is v is $Z=\rho\cdot v$. (1 mark)

For a displacement amplitude η_0 the energy flux density in the plane wave is $\frac{1}{2}\cdot\left(\eta_0\cdot\omega\right)^2\cdot Z$. (2 marks)

(Note - we may check dimensions: the impedance is $Z=\frac{pressure}{velocity}=\frac{M\cdot L\cdot T^{-2}\cdot L^{-2}}{L\cdot T^{-1}}=M\cdot L^{-2}\cdot T^{-1}$,

giving energy density $\left(M\cdot L^{-2}\cdot T^{-1}\right)\cdot\left(L\cdot T^{-1}\right)^2$ ie $\left(M\cdot L^2\cdot T^{-2}\right)\cdot L^{-2}\cdot T^{-1}=\frac{\left(\frac{energy}{area}\right)}{time}$

The reflected amplitude is $r=\frac{\left(Z_1-Z_2\right)}{\left(Z_1+Z_2\right)}$. (2 marks)

The incident energy flux density in the incident wave of unit amplitude is $\frac{1}{2}\cdot(\omega)^2\cdot Z_1$.

The sum of the reflected and transmitted energy flux densities is

$\frac{1}{2}\cdot\omega^2\cdot\left[Z_1\cdot\frac{\left(Z_1-Z_2\right)^2}{\left(Z_1+Z_2\right)^2}+Z_2\cdot\frac{\left(4\cdot Z_1^{\,2}\right)}{\left(Z_1+Z_2\right)^2}\right]=\frac{1}{2}\cdot\omega^2\cdot Z_1$, which is equal to the incident flux. (2 marks)

4. In a time t the source emits $f\cdot t$ cycles into a distance which is the distance travelled by the waves in that time minus the time travelled by the source in that time. That is, there are $f\cdot t$ cycles in a length $v\cdot t-v_s\cdot t$, giving a wavelength of $\frac{\left(v-v_s\right)\cdot t}{f\cdot t}$. As a result, the frequency observed will by the wave speed divided by this wavelength, that is

$\frac{f}{1-\frac{v_s}{v}}$. (3 marks)

If both source and observer are moving towards each other, the apparent frequency will be $\frac{\left(1+\frac{v_o}{v}\right)}{1-\frac{v_s}{v}}\cdot f$. (1 mark)

We know that $320=\frac{f}{1-\frac{v_s}{330}}$ and $280=\frac{f}{1+\frac{v_s}{330}}$, and eliminating the source frequency gives

$280\cdot\left(1+\frac{v_s}{330}\right)=320\cdot\left(1-\frac{v_s}{330}\right)$, or $40=(280+320)\cdot\frac{v_s}{330}$, from which we have $v_s=22$ metres/second. (4

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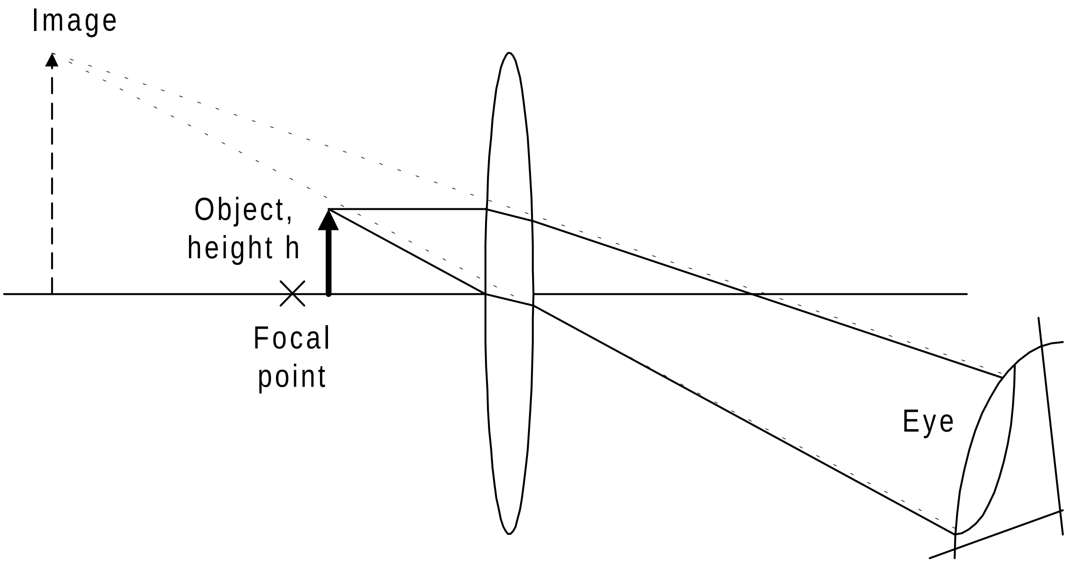
marks)

5. The ray diagram below shows all the relevant refractions.



(*3 marks*)

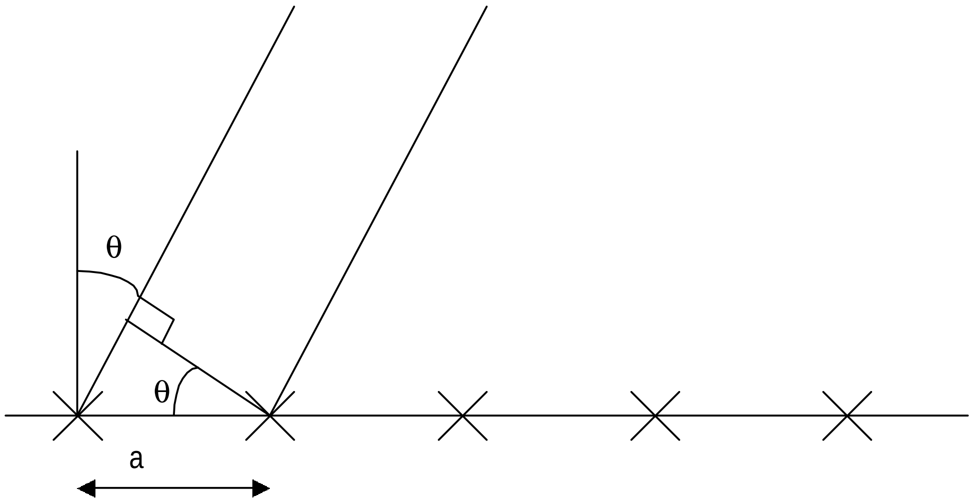
The magnification factor is the angular height of the image at infinity divided by the angular height of the object at the near point distance *d*. Let the height of the object be *h*. In the diagram above, if the image is at infinity the object is at the focal point of the lens, so the angular height of the image will be h/f. At the near point distance, on the other hand, the angular height of the object is h/d. Thus the magnification is $(h/f)/(h/d)$, that is $M = (d/f)$. Here, then, $f = (d/M) = (250/5) = 50$ mm. (4 marks)

6. The geometrical path difference between adjacent aerials for a beam at an angle θ is $a \cdot sin(\theta)$. If the signals from these aerials are to add in phase, this geometrical path difference must be an integral number of wavelengths. Hence the star's signal will be seen clearly if

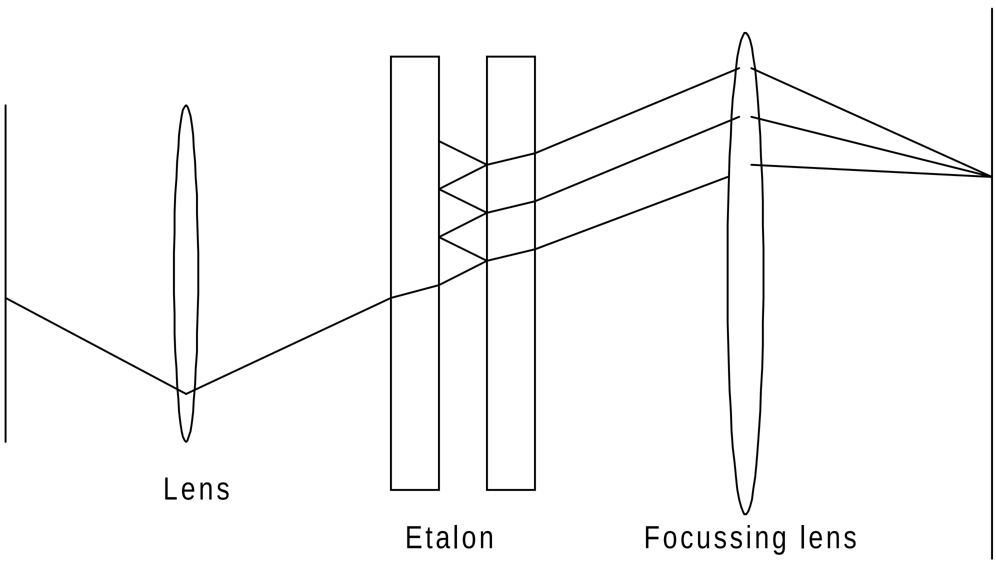
$a\ sin\left(\theta_n\right) = n \cdot \lambda$. (4 marks - a diagram, as below, is really necessary)

In the situation described, a = 7 m, λ = 210 10^{-3} m. The first minimum will occur at a path difference of one half wavelength, that is at an angle θ given by

$sin(\theta) = \dfrac{210 \cdot 10^{-3}}{2 \cdot 7} = 15 \cdot 10^{-3}$, or, using the approximation that sin(θ) is approximately equal to θ for small angles, θ is $\dfrac{210 \cdot 10^{-3}}{2 \cdot 7} \cdot \dfrac{180}{\pi} \cdot 60 = 52$ minutes of arc. N.B. 2 marks only out of 3 if angle calculated without giving units.

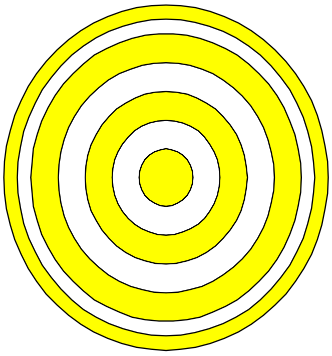


7. The Fabry-Perot interferometer:



3 marks

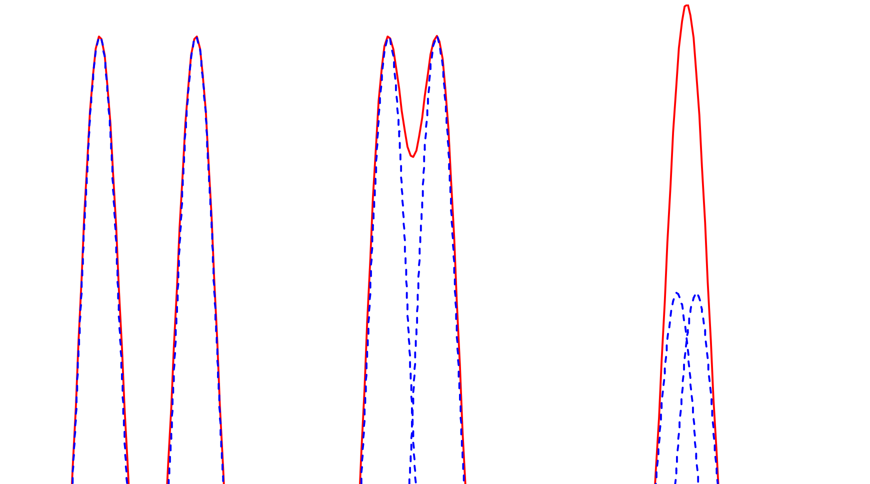
The pattern on the screen is formed by the interference of rays passing through the etalon at a particular angle, and the pattern is therefore formed of concentric rings. The central ring may be either dark or light, depending on the separation of the etalon.



1 mark

Note that the spacing of the rings increases as the centre of the pattern is approached - for full marks we do not require this, though.

The fringes will be visible if there is a clear dip in intensity between. The Rayleigh criterion expresses this by stating that two diffraction patterns are resolved if the maximum of one is on or beyond the first minimum of the other. In numerical terms, this leads to a dip in intensity to $8/\pi^2$ of the maximum. What is required here is not a numerical value, but a statement that the fringe widths should be small enough compared with the spacing that there is a visible separation, expressed in a diagram. 2 marks should be awarded for a statement of the Rayleigh criterion, 3 for relating it to the current situation.



8. The phase velocity is the rate of propagation of surfaces of constant phase in a wave, whereas the group velocity is the rate of propagation of a wave packet. Full marks should also be awarded (*pace* Brillouin) for defining the group velocity as the signal velocity or the rate of propagation of energy, or for definitions of phase and group velocities as ω/*k* and dω/d*k* respectively. (2 marks)

If the phase velocity $v_p = \frac{\omega}{k}$ then $\omega = k \cdot v_p$ and the group velocity $v_g = \frac{d\omega}{dk} = v_p + k \cdot \frac{dv_p}{dk}$. (2 marks)

For rollers, then, $v_p = \sqrt{\frac{g}{k}}$ so $v_g = \left(\sqrt{\frac{g}{k}} - k \cdot \frac{1}{2} \cdot \sqrt{\frac{g}{k^3}} \right) = \frac{1}{2} \cdot v_p$ which is less than the phase velocity, whereas for surface tension waves $v_p = \sqrt{\frac{\sigma \cdot k}{\rho}}$ so $v_g = \left(\sqrt{\frac{\sigma \cdot k}{\rho}} + k \cdot \frac{1}{2} \cdot \sqrt{\frac{\sigma}{\rho \cdot k}} \right) = \frac{3}{2} \cdot v_p$, which is greater than the phase velocity. (3 marks)