Part B

9. Consider a rod with cross-sectional area A. Let the longitudinal displacement of a plane perpendicular to the axis of the rod at a point x along its length be u(x), and that at x+dx be u(x)+du(x)=u(x)+(du(x)/dx)dx. Thus the extension of the section of length dx is du, so the force on the cross-section at x will be, by Hooke's law, equal to Young's modulus times the cross-sectional area times the strain at x, or

$$F(x) = Y\cdot A\cdot\left(\frac{du}{dx}\right).$$

The force at x+dx, however, will be F+(dF/dx) dx, so the nett force on the element of length dx will be

$$dF = Y\cdot A\cdot\frac{d^2}{dx^2}u\cdot dx$$

and as the mass of the element is $\rho\cdot A\cdot dx$ Newton's law gives

$$\frac{d^2}{dt^2}u = \frac{Y}{\rho}\cdot\frac{d^2}{dx^2}u$$

which (noting that the derivatives should be partials - a deficiency of this program) is the one-dimensional wave equation with wave speed $v = \sqrt{\frac{Y}{\rho}}$. (5 marks)

At a join, at x=0, say, between two rods both the displacement u and the force F must be continuous. (1 mark)

Thus if we consider a wave travelling to the right in material 1, on the left, of unit amplitude, giving rise to a transmitted wave of amplitude t in material 2, on the right, and a reflected wave of amplitude r, we have

in medium 1 a displacement: $e^{i\cdot(\omega\cdot t - k_1\cdot x)} + r\cdot e^{i\cdot(\omega\cdot t + k_1\cdot x)}$

and a force per unit area $:Y_1\cdot\frac{du}{dx} = Y_1\cdot\left[-k_1\cdot e^{i\cdot(\omega\cdot t - k_1\cdot x)} + r\cdot k_1\cdot e^{i\cdot(\omega\cdot t + k_1\cdot x)}\right]$

whereas we have

in medium 2 a displacement: $t\cdot e^{i\cdot(\omega\cdot t - k_1\cdot x)}$

and a force per unit area $:Y_2\cdot\frac{du}{dx} = Y_2\cdot\left[-t\cdot k_2\cdot e^{i\cdot(\omega\cdot t - k_2\cdot x)}\right]$.

Equating the displacements and forces at x=0 gives

$1 + r = t$ and

$Y_1\cdot k_1\cdot(r-1) = -Y_2\cdot k_2\cdot t.$

Eliminating r gives $2 = \left(1 + \frac{Y_2\cdot k_2}{Y_1\cdot k_1}\right)\cdot t$ or $t = \frac{2\cdot Y_1\cdot k_1}{Y_1\cdot k_1 + Y_2\cdot k_2}$.

We may eliminate the common factor of the frequency to obtain

$t = \frac{2\cdot\sqrt{Y_1\cdot\rho_1}}{\sqrt{Y_1\cdot\rho_1} + \sqrt{Y_2\cdot\rho_2}}$ and further $r = \frac{\sqrt{Y_1\cdot\rho_1} - \sqrt{Y_2\cdot\rho_2}}{\sqrt{Y_1\cdot\rho_1} + \sqrt{Y_2\cdot\rho_2}}$. (4 marks)

For the three-material system, take the joins to be at x= 0 (join 1 to m) and at x = l (join m to 2). The same continuity equations lead to 4 equations in the 4 unknowns (reflection amplitude in medium 1, transmitted amplitude in medium 2, and left and right-travelling waves A and B in medium m):

$$1 + r = B + A \qquad (1)$$

$$B\cdot e^{-i\cdot k_m\cdot l} + A\, e^{i\cdot k_m\cdot l} = t\cdot e^{-i\cdot k_2\cdot l} \qquad (2)$$

$$-k_1\cdot Y_1 + r\cdot k_1\cdot Y_1 = \left(-B\cdot k_m\cdot Y_m\right) + A\cdot k_m\cdot Y_m \qquad (3)$$

$$-B\cdot k_m\cdot Y_m\cdot e^{-i\cdot k_m\cdot l} + A\cdot k_m\cdot Y_m\cdot e^{i\cdot k_m\cdot l} = -t\cdot k_2\cdot Y_2\cdot e^{-i\cdot k_2\cdot l} \qquad (4)$$

which are best simplified by first eliminating t by multiplying equation (2) by $k_2\cdot Y_2$ and adding it to equation (4), giving

$$B\cdot\left(k_2\cdot Y_2 - k_m\cdot Y_m\right)\cdot e^{-i\cdot k_m\cdot l} + A\cdot\left(k_2\cdot Y_2 + k_m\cdot Y_m\right)\cdot e^{i\cdot k_m\cdot l} = 0. \qquad (5)$$

Now we note that if there is to be no reflection (r=0) we must have

$$A + B = 1$$

and

$$A - B = \frac{k_1\cdot Y_1}{k_2\cdot Y_2}$$

so that

$$A = \frac{1}{2} + \frac{k_1\cdot Y_1}{2\cdot k_2\cdot Y_2} \quad \text{and} \quad B = \frac{1}{2} - \frac{k_1\cdot Y_1}{2\cdot k_2\cdot Y_2}.$$

Substituting these values into (5), and rearranging, we find

$$\left(k_2\cdot Y_2 - k_1\cdot Y_1\right)\cdot\cos\left(k_m\cdot l\right) + i\cdot\frac{\left[\left(k_m\cdot Y_m\right)^2 - k_1\cdot k_2\cdot Y_1\cdot Y_2\right]}{k_m\cdot Y_m}\cdot\sin\left(k_m\cdot l\right) = 0.$$

Now we know from above that if there would have been a reflected signal from the simple join between materials 1 and 2 then $k_2\cdot Y_2 - k_1\cdot Y_1$ is not zero. Thus the thickness of the matching layer must be such as will make the cosine zero -- the thinnest such layer has a thickness of one quarter of the wavelength in the matching layer. Note that the question does not ask for the impedance of the matching layer. (10 marks)