PHYS1B24 Model Answers

1. If the pressure is

$$p(x,t) := 2\cdot\cos\left(k_1\cdot x - \omega_1\cdot t\right)\cdot\sin\left(k_2\cdot x - \omega_2\cdot t\right)$$

we may use a standard trigonometric relationship to rewrite it as

$$p(x,t) := \sin\left[\left(k_1 + k_2\right)\cdot x - \left(\omega_1 + \omega_2\right)\cdot t\right] + \sin\left[\left(k_2 - k_1\right)\cdot x - \left(\omega_2 - \omega_1\right)\cdot t\right],$$

which shows that it is a superposition of two waves, of frequencies $(\omega_1+\omega_2)/2\pi$ and $(\omega_1-\omega_2)/2\pi$, in this case

$$\frac{561\cdot\pi + 33\cdot\pi}{2\cdot\pi} = 297 \quad \text{Hz}$$

and

$$\frac{561\cdot\pi - 33\cdot\pi}{2\cdot\pi} = 264 \quad \text{Hz.}$$

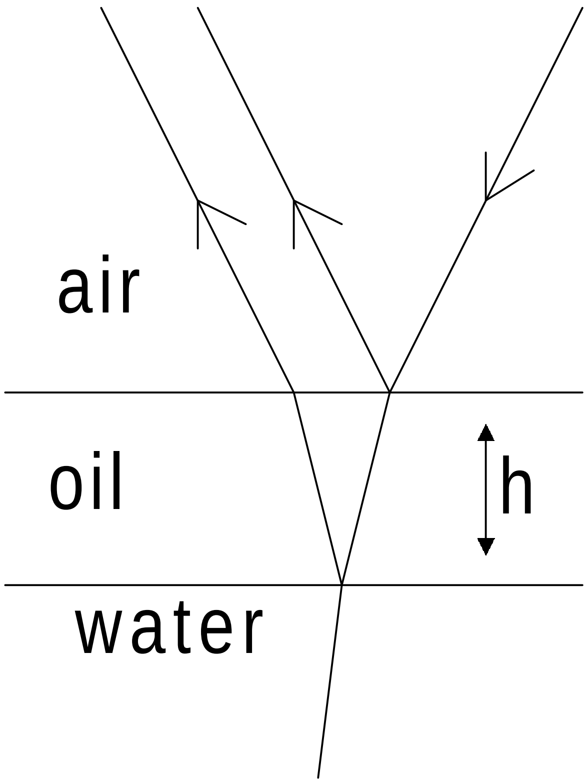
The speed of sound in air may be deduced from the wave speed of the separate sinusoidal waves, or from either component of the product form of the wave, using the formula $c := \frac{\omega}{k}$, thus

$c := \frac{\omega_1}{k_1}$ gives $\frac{33\cdot\pi}{0.1\cdot\pi} = 330$ m s^{-1} $\qquad$ $c := \frac{\omega_2}{k_2}$ gives $\frac{561\cdot\pi}{1.7\cdot\pi} = 330$ m s^{-1}

$c := \frac{\omega_1 + \omega_2}{k_1 + k_2}$ gives $\frac{33\cdot\pi + 561\cdot\pi}{0.1\cdot\pi + 1.7\cdot\pi} = 330$ m s^{-1} and

$c := \frac{\omega_1 - \omega_2}{k_1 - k_2}$ gives $\frac{33\cdot\pi - 561\cdot\pi}{0.1\cdot\pi - 1.7\cdot\pi} = 330$ m s^{-1} .

2. The interference is caused by interference between light reflected from the top of the oil film and light which has entered the oil and been reflected from the oil-water interface.



The phase change in the reflection in air from the upper surface of the oil is π (whereas the phase change on reflection at the oil-water interface is 0).

Constructive interference takes place when the total phase difference between a ray reflected from the surface of the oil and a ray which has passed into the oil and been reflected from the water is an integer multiple of 2π. Given the phase changes on reflection differ by π for the two rays, this requires the path length in oil to be an odd half-integer multiple of the wavelength *in the oil*. The film is observed at near-normal incidence, so we may take the geometrical path length in the oil to be 2h.We therefore seek a wavelength λ in air which satisfies $2\cdot h := \left(p + \frac{1}{2}\right)\cdot\frac{\lambda}{n_{oil}}$ or

$\lambda := \frac{4\cdot h\cdot n_{oil}}{2\cdot p + 1}$ where *p* is an integer, or $\lambda := \frac{2\cdot 0.27\cdot 1.44}{p + \frac{1}{2}}$ µm, i.e. if $p := 0, \frac{2\cdot 0.27\cdot 1.44}{p + \frac{1}{2}} = 1.555$ µm, if

$p := 1, \frac{2\cdot 0.27\cdot 1.44}{p + \frac{1}{2}} = 0.518$ µm, and if $p := 2, \frac{2\cdot 0.27\cdot 1.44}{p + \frac{1}{2}} = 0.311$ Thus only the first order interference falls in the visible, with a wavelength of 0.52 µm.

3. The velocity of compression waves in a gas is given by

$$v := \sqrt{\frac{B}{\rho}}$$

where *B* is the bulk modulus and ρ is the density.

The adiabatic bulk modulus is appropriate in most cases because the thermal conductivity of the gas is normally so low that there is not time in one period of the wave for the temperature changes produced by compression and rarefaction of the gas to be smoothed out by conduction.

The amplitude transmission from medium 1 to medium 2 is given by

$$t := \frac{2 \cdot Z_1}{Z_1 + Z_2}$$

If Z_2 is very large, the transmission will be very small. The impedance measures the generalized force (here, pressure) required to produce a unit response (here, particle velocity), and so the response will be very small in a material of much higher impedance than that from which the wave is incident.

In the case in question, the two gases are at the same pressure, so it is not necessary to know what the pressure is because it cancels out of the equation for *t*. Thus

$$t := \frac{2 \cdot \sqrt{\gamma_{air} \cdot p \cdot \rho_{air}}}{\sqrt{\gamma_{air} \cdot p \cdot \rho_{air}} + \sqrt{\gamma_{He} \cdot p \cdot \rho_{He}}}$$

which gives

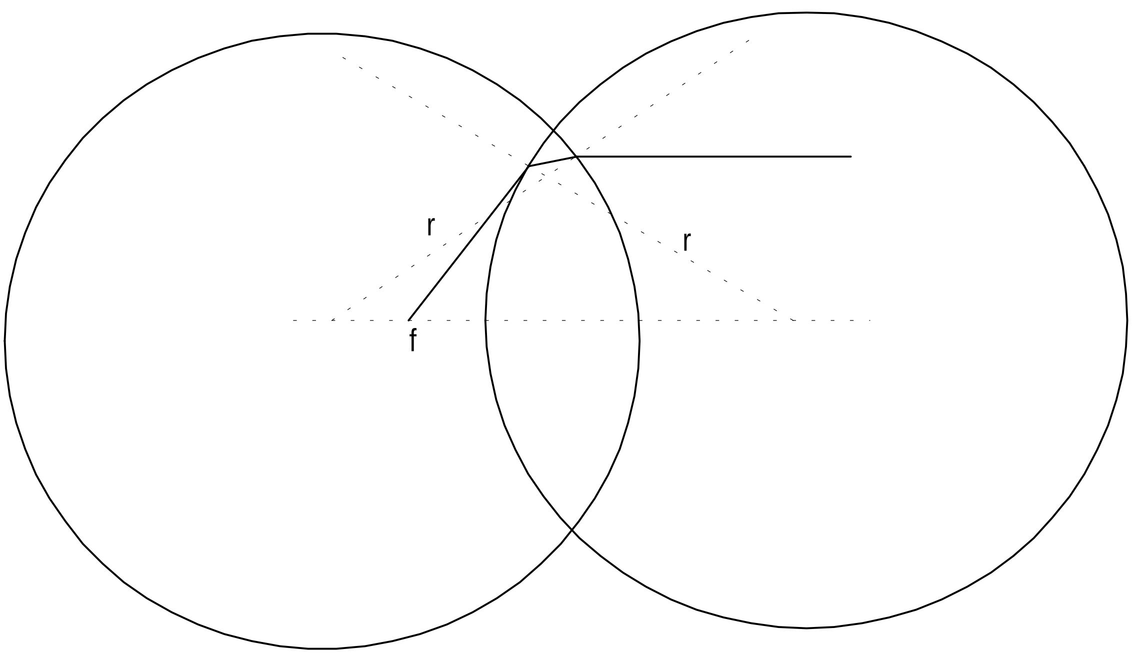
$$\frac{2 \cdot \sqrt{\frac{7}{5} \cdot 1.18}}{\sqrt{\frac{7}{5} \cdot 1.18} + \sqrt{\frac{5}{3} \cdot 0.18}} = 1.402$$

4. A bispherical biconvex lens formed from a material of refractive index n with radii of curvature r_1 and r_2 embedded in a medium of refractive index 1 has a focal length f given by

$$\frac{1}{f} := (n-1)\cdot\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

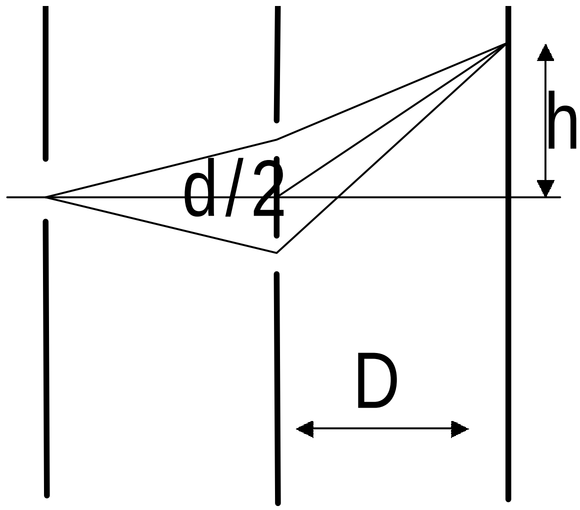
which uses the convention that the radius of curvature is positive if the ray encounters it from the convex side, negative if the ray encounters it from the concave side, and the ray encounters surface 1 first.

If we are to produce a parallel beam, we place the source at the focus, so we are being told that the focal length is $f := 2$ m, and $n := 12$. The radius of curvature, then, is given by $r := 2\cdot(n-1)\cdot f$, or $r = 44$ m.



The key points that the diagram should show (and to do so it will almost certainly not be to scale - in fact an enlargement of a region showing just the passage of the ray through the lens would be an intelligent solution) is the bending of the ray towards the normal as the light enters the lens and away from the normal as it leaves.

5. The geometry of this (Young's slits) experiment is shown below.



The light and dark bands are formed by interference between light from the two slits, and positive interference (light bands) occurs when the path lengths differ by an integral number of wavelengths in the medium between the slits and the screen.

The path length from the upper slit to the screen a distance h from the centre line is

$s_1 := \sqrt{D^2 + \left(h - \frac{d}{2}\right)^2}$ and the path length from the lower slit is $s_2 := \sqrt{D^2 + \left(h + \frac{d}{2}\right)^2}$

Thus the geometric path difference is

$$s_2 - s_1 := \sqrt{D^2 + \left(h + \frac{d}{2}\right)^2} - \sqrt{D^2 + \left(h - \frac{d}{2}\right)^2}$$

Expanding the square roots we have the approximate result

$$\Delta s := \frac{h \cdot d}{D}$$

Thus bright bands will be seen on the screen at

$h := p \cdot \lambda \cdot \frac{D}{d}$ where *p* is an integer and the spacing between successive bright bands is $h := \frac{\lambda D}{d}$

Full marks will be given if this result is quoted without derivation.

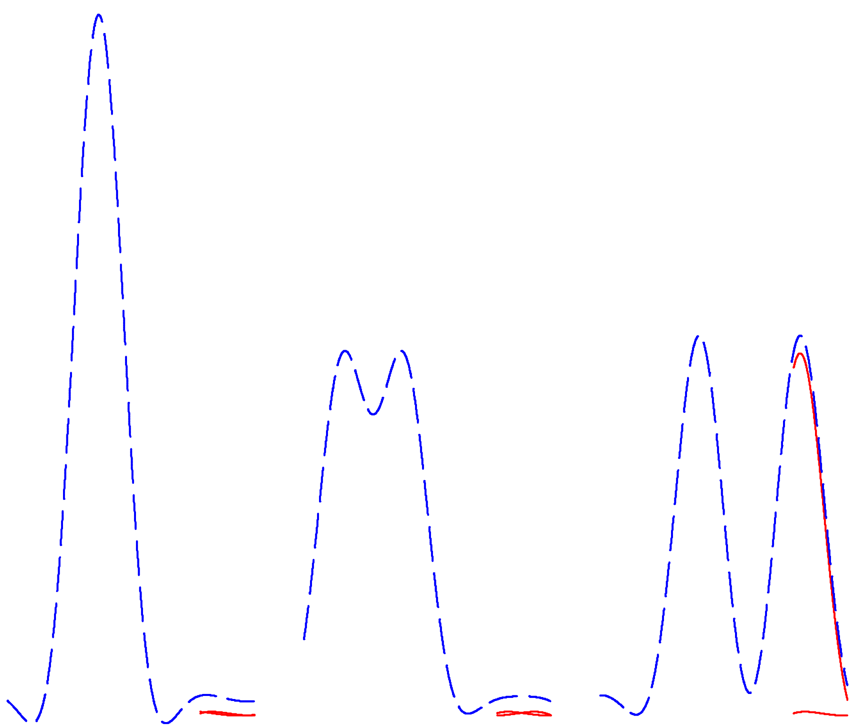
If the space between the slits and the screen is filled with a medium of refractive index *n*, the wavelength of the light will be altered to λ/n, and the spacing of the fringes will therefore be changed to

$\Delta h := \frac{\lambda}{n} \cdot \frac{D}{d}$, that is, if $n>1$ the fringe spacing will be reduced.

6. Rayleigh's criterion for the resolution of the images of two slit sources is based on the idea that there must be a significant dip in intensity between the two peaks. The dip required is specified, somewhat arbitrarily, as being that which occurs when the first zero of one image coincides with the central maximum of the other (see diagram below). Now the angular separation between the first minimum and the centre of the diffraction pattern of a slit of width *d* is θ = sin-1(λ/*d*) or, to a good approximation, θ = λ/*d*, and so two slits of identical width *d* will be resolved if their angular separation is λ/*d*.

For two identical point sources of diameter *d* the corresponding condition is θ = 1.22 λ/*d*. (It is the case, but need not be stated, that this is related to the first zero of a Bessel function of order 1).

$$I(\theta), I(\theta+\phi), I(\theta)+I(\theta+\phi), \left(\frac{\sin(ka\cdot\sin(\theta))}{ka\cdot\sin(\theta)}\right)^2, \left(\frac{\sin(ka\cdot\sin(\theta+\phi))}{ka\cdot\sin(\theta+\phi)}\right)^2, \left(\frac{\sin(ka\cdot\sin(\theta))}{ka\cdot\sin(\theta)}\right)^2 + \left(\frac{\sin(ka\cdot\ldots}{ka\cdot\sin\ldots}\right.$$



Traces not separated, at Rayleigh separation, well above Rayleigh separation.

In the case of the reflecting telescope, the resolution is governed by the diameter of the objective. From the Rayleigh criterion, the minimum angular separation that can be distinguished with a wavelength $\lambda := 500\cdot 10^{-9}$ m and a mirror of diameter $d := 0.2$ m is given by $\theta := 1.22\cdot\frac{\lambda}{d}$ or $\theta = 3.05\cdot 10^{-6}$ radians (which may be expressed as

$\theta\cdot\frac{180}{\pi}\cdot 60\cdot 60 = 0.629$ i.e. 0.6 seconds of arc).

7. The phase velocity is the speed at which planes of equal phase, for example maxima or minima of displacement in a pressure wave, travel through the medium: it is given by $\frac{\omega}{k}$.

The group velocity is the speed at which energy is transmitted through the medium, or the signal velocity in the medium, or the velocity at which the envelope of a wave packet travels.

Any of these definitions, or simply $u := \frac{d\omega}{dk}$, will be acceptable.

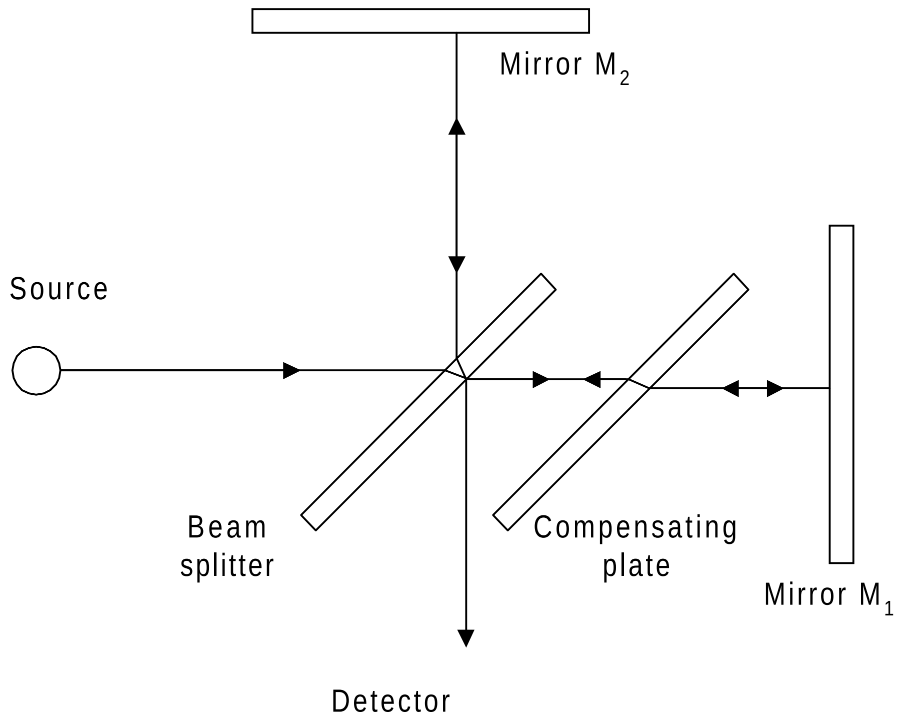
If we write $\omega := v_p \cdot k$ and differentiate with respect to *k* we obtain

$$\frac{d\omega}{dk} := k \cdot \frac{dv_p}{dk} + v_p = v_g$$

Then, if $v_p := A + \frac{B}{k^2}$

$$v_g := k \cdot \frac{-2 \cdot B}{k^3} + \left(A + \frac{B}{k^2}\right) = A - \frac{B}{k^2}$$

8.

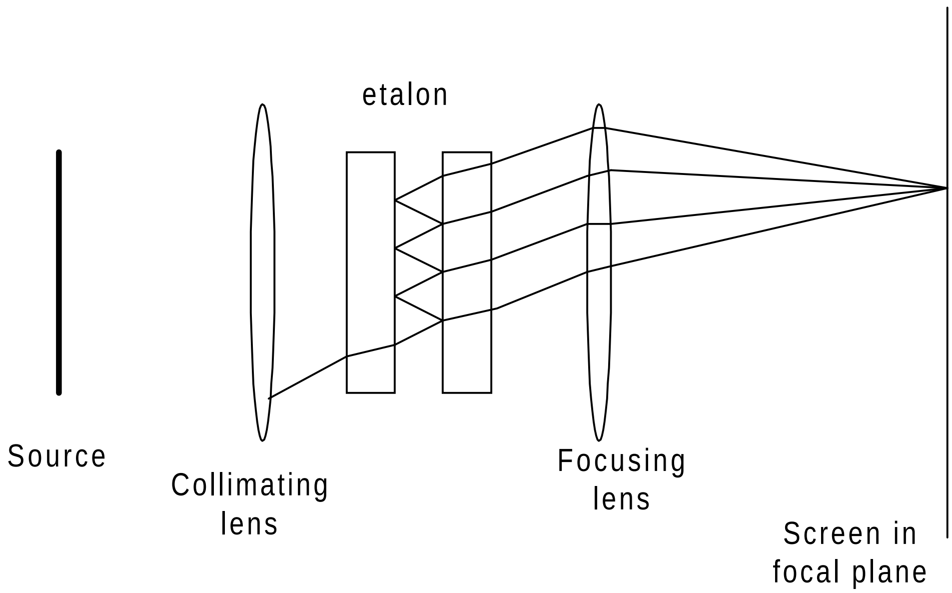


As drawn, the mirrors M_1 and M_2 are silvered on their front faces, and the beam splitter consists of a glass plate part-silvered on its rear surface.

Circular fringes will be observed at the detector if the reflection of M_1 in the beam splitter is parallel to M_2.

The glass in the beam splitter will be dispersive. This plate, however, is traversed thrice by the beam which traverses the path (source-beam splitter-M_2-beam splitter-detector) but only once by the beam (source-beam splitter-M_1-beam splitter-detector). This results in a wave-length dependent optical path difference. If the compensator plate is identical (apart from any silvering) to the beam splitter, and is aligned parallel to it, it will introduce an extra path length into the second path which is exactly equivalent to the two extra traverses of the beam splitter in the first path.

9. The form of interferometer which the student will probably describe is the original Fabry-Perot form, but full credit will also be given for a description of variants such as the scanning interferometer or the Connes (spherical) variant. The diagram should show both the collimating and focusing lenses, the etalon itself (it is not essential to show either the angling of the outer faces or the silvering of the inner ones, although the diagram should imply the latter) and the screen.



The sketch of the pattern on the screen should show a series of concentric bright circles, the spacing decreasing as the radius increases.

If the amplitude transmission coefficient from air through one plate of the etalon into the air gap is *t*, the reflection coefficient at each inner surface of the etalon is *r*, and the transmission through the second plate is *u*, the phase shift in a to-and-fro reflection between the plates is δ (which we take to include the phase change on reflection, so that *r* is real), then the total transmitted amplitude from unit incident amplitude is

$$A := t\cdot u + t\cdot r\cdot r\cdot e^{i\cdot\delta}\cdot u + t\cdot r\cdot r\cdot e^{i\cdot\delta}\cdot r\cdot r\cdot e^{i\cdot\delta}\cdot u + \ldots$$

which is an infinite geometric progression with ratio $r^2\cdot e^{i\cdot\delta}$, so

$$A := \frac{t\cdot u}{1 - r^2\cdot e^{i\cdot\delta}}.$$

The transmitted intensity is the product of *A* and its complex conjugate,

$$I := \frac{(|t\cdot u|)^2}{1 + r^4 - 2\cdot r^2\cdot\cos(\delta)} = \frac{(|t\cdot u|)^2}{(1 - r^2)^2}\cdot\frac{1}{1 + \frac{4\cdot r^2}{(1 - r^2)^2}\cdot\sin\left(\frac{\delta}{2}\right)^2}$$

Peaks occur when the phase difference δ is an integer multiple of 2π, when the sine term in the denominator of the above expression is zero. The half-width of the peak is thus given by 2ε denominator of the above expression is zero. The half-width of the peak is thus given by 2ε,

where ε is the angle at which the intensity falls to half its peak value. Thus

$$\frac{4\cdot r^2}{\left(1-r^2\right)^2}\cdot \sin\left(\frac{\varepsilon}{2}\right)^2 := 1 \quad ,$$

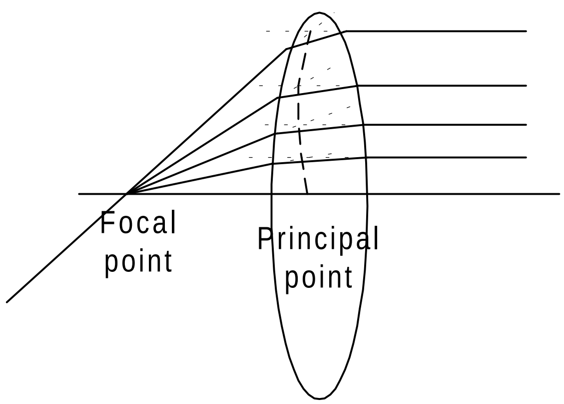
and as ε will be small

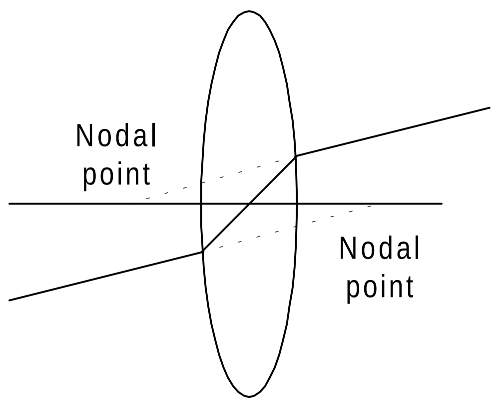
$$2\cdot\varepsilon := 2\cdot\frac{1-r^2}{r} \quad .$$

A stable interference pattern is formed with an extended source because the interference occurs as a result of interference between the rays formed within the etalon by partial reflection of **every** ray from the source. Rays from two points on the source which enter the system at equal angles will arrive at the same point on the screen, but each one will individually give rise to a maximum or minimum of intensity. If the interference is destructive, each point gives zero intensity at the screen. If the interference is constructive, each point gives rise to a large amplitude at the screen: as the two points on the source will be uncorrelated, the result at the screen will be the sum of the intensities.

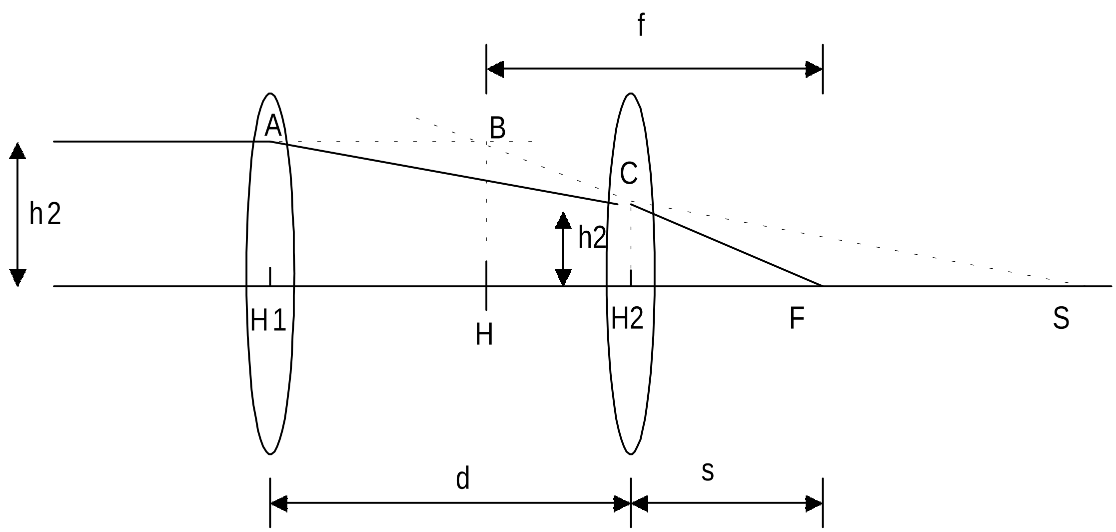
The resolution of the etalon depends on the reflectivity (as demonstrated above in the calculation of the half-width), so high reflectivity is favoured. However, if this is achieved by silvering the faces, the transmission is reduced (the factors t and u). With dielectric films, it is possible to achieve reflectivities close to unity with negligible loss.

10. The focal points of the lens system are the points at which a ray (or its projection) incident parallel to the axis of the lens system crosses the axis after passing through the system. The principal points of the lens system are the points at which the principal planes intersect the axis. The principal plane is the planar approximation to the surface formed by the locus of the points of intersection of the rays parallel to the axis and the corresponding rays refracted through the focal points. The nodal points are the points of intersection with the axis of rays which pass through the optical centre of the system, that is, rays which pass through the lens without being deviated in angle.
In the diagrams below, only one set of the focal and principal cardinal points is shown: a corresponding diagram for rays from the other side (which need not be given) will show the other set.





The properties of the two-lens system may be computed using the figure shown below.



The incoming rays from the left are focused by lens 1 at the point S. This forms the virtual source for lens 2, which is thus a distance (f_1-d) to the right of lens 2. Then lens 2 will form an image a distance H2-F = s from lens 2, where

$$\frac{1}{f_2} := \frac{-1}{(f_1 - d)} + \frac{1}{s} \qquad (1)$$

(note the use of the sign convention)

If the initial ray enters lens 1 at a height h_1, and the ray exits the system at a height h_2 in lens 2, then the intersection point (giving the position of the principal point H) is given by equating the value of h_2 found in two ways (using the properties of similar triangles C F H2 and B F H, C S H2 and A S H1):

$$h_2 := \frac{h_1}{f}\cdot s = \frac{h_1}{f_1}\cdot\left(f_1 - d\right)$$

from which we may find *s* in terms of known quantities, and substitute back into (1) to find

$$\frac{1}{f_2} := \frac{-1}{f_1 - d} + \frac{f_1}{f\cdot\left(f_1 - d\right)}$$

which may be rearranged to give

$$\frac{1}{f} := \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1\cdot f_2}$$ as required.

The same equations allow us to locate the positions of the principal points: from above we have

$$s = \frac{f}{f_1}\cdot\left(f_1 - d\right)$$

whence we obtain the distance from the principal point H to the lens

$$s - f = \frac{f}{f_1}\cdot\left(f_1 - d\right) - f = -\frac{f\cdot d}{f_1}$$ (note the sign, denoting the fact that in the case in which both lenses are converging the principal point we have just found, the second principal point, lies to the left of the second lens).

The corresponding formula for the first principal point gives its position relative to the first lens as $\frac{f\cdot d}{f_2}$.

With the figures given, we have for the focal length of the system

$$\frac{1}{f} = \frac{1}{60} + \frac{1}{30} - \frac{40}{30\cdot 60} = \frac{5}{180}$$ so the focal length is 36mm.

Hence we may find the position of the first principal point, which is

$$\frac{36\cdot 40}{30} = 48$$ mm from the first lens, and of the second principal point, which is

$$\frac{36\cdot 40}{60} = 24$$ mm from the second lens. Putting these together, we have (if we place the 60mm lens at x=0 and the 30mm lens at x=40,

first principal point at x=48mm
first focal point at x=48-36 = 12mm
second principal point at x=40-24=16mm
second focus at x=16+36=52mm

11. Huygens's principle states that every point on a propagating primary wavefront acts as a source of spherical wavelets, so that the wavefront at some later time is the envelope of these wavelets. The wavelets advance with the same speed and have the same frequency as the primary wave.

To account for diffraction, Huygens's concentration on the envelope has to be extended (the Huygens-Fresnel principle) so that the amplitude at any point is the superposition of all the wavelets, taking account of their amplitudes and phases. Diffraction may be treated in the Fraunhofer approximation when both the source and the point of observation are far enough away from the diffracting object for the incident and diffracted waves to be treated as planar. In this regime one may make the following approximations:

1. The optical path length from the source to the object may be taken to be a constant;
2. The range dependence of the amplitude of the spherical wavelets (1/r) may be taken to be constant
3. The obliquity factor may be taken as constant.

(Point 3 may be omitted, as the obliquity factor does not occur in Huygens's model).

The complex amplitude may then be written as

$$\psi := (\text{constant})\cdot\int\int e^{-i\cdot k\cdot r}\,dx\,dy$$

where the integral is taken over the diffracting aperture, *k* is the wavevector, and *r* is the distance from the point (*x*,*y*) in the aperture to the point of observation.

If the observation direction is an angle θ away from the normal to the grating, then the path difference between one slit and the next is $h\sin\theta$, giving a phase difference between adjacent slits of

$$\frac{2\cdot\pi}{\lambda}\cdot h\cdot\sin(\theta)=2\cdot\gamma\ .$$

If *a* is the amplitude due to the first slit at the point of observation, then the total amplitude may be written as

$$A := a + a\cdot e^{-2\cdot i\cdot\gamma} + a\cdot e^{-4\cdot i\cdot\gamma} + \ldots\ldots + a\cdot e^{(1-N)\cdot 2\cdot i\cdot\gamma}$$

Now we may perform the summation as a geometric progression of *N* terms with first term a and common ratio $e^{-2\cdot i\cdot\gamma}$ The result is

$$A := a\cdot\frac{1-e^{-2\cdot i\cdot N\cdot\gamma}}{1-e^{-2\cdot i\cdot\gamma}}=a\cdot\frac{e^{-i\cdot N\cdot\gamma}}{e^{-i\cdot\gamma}}\cdot\frac{e^{i\cdot N\cdot\gamma}-e^{-i\cdot N\cdot\gamma}}{e^{i\cdot\gamma}-e^{-i\cdot\gamma}}$$

We can also simplify this to

$$A := a\cdot e^{i\cdot(1-N)\cdot\gamma}\cdot\frac{\sin(N\cdot\gamma)}{\sin(\gamma)}$$

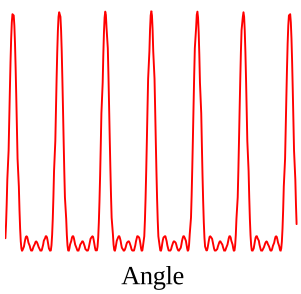
The intensity *I*(θ) is proportional to the square of the modulus of *A*, and as the limiting form of the modulus of *A* as θ tends to zero is *N*|*a*|, it follows that

$$I(\theta) := I(0)\cdot\left(\frac{\sin(N\cdot\gamma)}{N\cdot\sin(\gamma)}\right)^2\ .$$

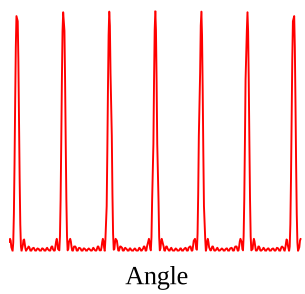
The sketches should demonstrate

a) that as the number of slits increases, the spacing of the principal maxima increases;

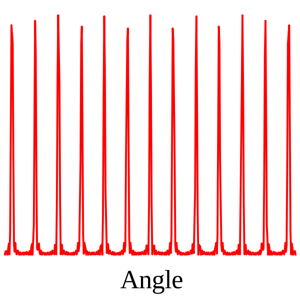
b) that as the slits broaden, the envelope of the principal maxima gets narrower.



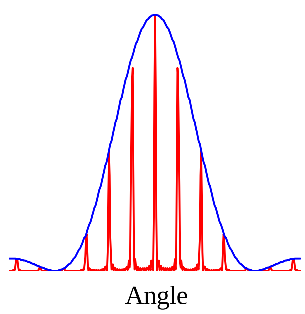
5 narrow slits



10 narrow slits

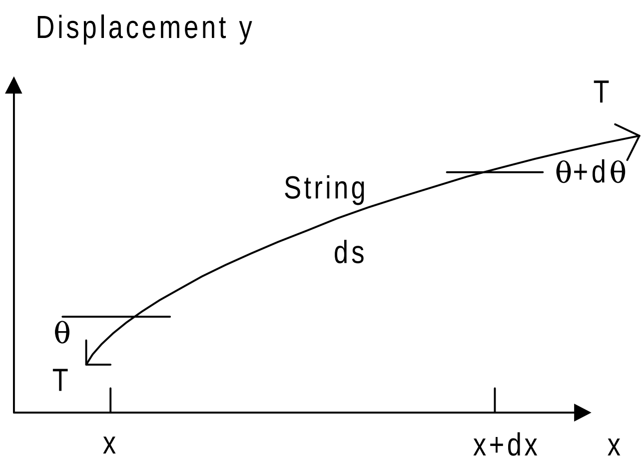


10 narrow slits



10 wide slits

12. Consider the string pulled aside as shown in the figure



The forces acting on the length ds of the string are the tension T at an angle θ at x and the same tension T at an angle θ+dθ at x+dx. If the displacement is small, so that the gradient of the string dy/dy is small, the length dx is approximately equal to ds. Also, the angle θ will be small, so that sin(θ) and tan(θ) are approximately equal, and both may be represented by dy/dx.

The vertical force on the string element, then, is given by

$$dF := T\cdot\left[\left(\frac{dy}{dx}\right)_{x+dx} - \left(\frac{dy}{dx}\right)_{x}\right] = T\cdot\left(\frac{d^2}{dx^2}y\right)\cdot dx$$

The mass of this element of the string, however, is ρdx, and we therefore have the equation of motion of the string element as

$$T\cdot\left(\frac{d^2}{dx^2}y\right)\cdot dx = \rho\cdot\left(\frac{d^2}{dt^2}y\right)\cdot dx$$

or

$y_{xx} = \frac{1}{v^2}\cdot y_{tt}$, where $v^2 = \frac{T}{\rho}$.

If the string is held fixed at the ends *x=0* and *x=L*, the displacement

$$y(x,t) := \mathrm{Re}\left[a\cdot e^{i\cdot(\omega\cdot t - k\cdot x)} + b\cdot e^{i\cdot(\omega\cdot t + k\cdot x)}\right]$$

where *a* and *b* may in general be complex, must be zero at *x=0* and *x=L* for all *t*. Thus *y* must in general be of the form (with ϕ a phase shift to be determined from the initial conditions)

$$y(x,t) := A\cdot\cos(\omega\cdot t + \phi)\cdot\sin\left(k_n\cdot x\right)$$

with $k_n := \frac{n \cdot \pi}{L}$ where *n* is any integer greater than zero. As we know that $v := \frac{\omega}{k}$ this means that the allowed frequencies have the form

$$f_n = \frac{\omega_n}{2 \cdot \pi} = \frac{v \cdot k_n}{2 \cdot \pi} = \frac{n \cdot v \cdot \pi}{2 \cdot \pi \cdot L} = \frac{n}{2 \cdot L} \cdot \sqrt{\frac{T}{\rho}}$$

The kinetic energy of the element of the string between *x* and *x+dx* is

$$dK := \frac{1}{2} \cdot \rho \cdot \left(\frac{dy(x,t)}{dt}\right)^2 \cdot dx = \frac{1}{2} \cdot \rho \cdot A^2 \cdot \omega_1^{\,2} \cdot \sin(\omega_1 \cdot t + \phi)^2 \cdot \sin(k_1 \cdot x)^2 \cdot dx$$

so the total kinetic energy is the integral of this from *x=0* to *L*, that is,

$$K := \frac{L}{4} \cdot \rho \cdot A^2 \cdot \omega_1^{\,2} \cdot \sin(\omega_1 \cdot t + \phi)^2 .$$

Now if the maximum value of the potential energy is equal to the maximum value of the kinetic energy but the total energy is constant it follows that the potential energy varies in quadrature with the kinetic energy, and that the total energy is

$$E := \frac{L}{4} \cdot \rho \cdot A^2 \cdot \omega_1^{\,2} = \frac{L}{4} \cdot \rho \cdot A^2 \cdot \left(\frac{2 \cdot \pi}{2 \cdot L} \cdot \sqrt{\frac{T}{\rho}}\right)^2 = \frac{A^2 \cdot T \cdot \pi^2}{4 \cdot L}$$ in the fundamental mode.

If the frequency with which the string is vibrating is 400Hz, and this is the fundamental frequency, we can deduce that the length of the string is given by

$L := \frac{1}{2 \cdot f} \cdot \sqrt{\frac{T}{\rho}}$ or $\frac{1}{2 \cdot 400} \cdot \sqrt{\frac{5}{2.5 \cdot 10^{-3}}} = 0.056$ m. Thus the total energy stored in the string if the amplitude is 10 mm is

$$\frac{.056}{4} \cdot 2.5 \cdot 10^{-3} \cdot (10 \cdot 10^{-3})^2 \cdot (2 \cdot \pi \cdot 400)^2 = 0.022$$ Joules.

13. In the non-relativistic case, we treat the effects of motion of the source and of the observer in inequivalent ways. For the motion of the source, the waves filling the space between the source and the observer will be stretched to fill a larger region of space than they would have had the source been stationary. In a time interval *t* the *ft* waves which are emitted will be stretched from a wavelength λ to a wavelength λ_1, where

$$f \cdot t \cdot \lambda + v_s \cdot t := f \cdot t \cdot \lambda_1$$

which converts into a frequency seen by an observer of

$$f_1 := \frac{f \cdot c^1}{c^1 + v_s} .$$

This result will be appropriate to an observer in another medium, as can readily be seen as we can imagine a stationary 'repeater' in the medium c^1 who simply reradiates at the received frequency to another observer - or, more simply, transmitting a wave from one medium to another does not alter its frequency.
For the moving observer, however, the rate at which the crests and troughs reach the observer is reduced from f to $f\,(c-v_o)/c$. Combining these two expressions, we have

$$f_{obs} := f \cdot \left(\frac{c - v_o}{c^1 + v_s} \right) \cdot \frac{c^1}{c}$$

In the case of the signal rocket, the beat frequency of 10 Hz coupled with the rocket velocity of 600 m/s tells us that the velocity of electromagnetic waves near the source (the rocket) is given by

$$c^1 := \frac{v_s \cdot f_{obs}}{f - f_{obs}} \quad \text{or}$$

$\dfrac{600 \cdot 10^7}{10} = 6 \cdot 10^8$ m/s, from which we deduce that the refractive index of the ionosphere near the rocket at a frequency of 10 MHz is 1/2.

In free space our expression for the observed frequency is (expanding the bottom line)

$$f_{obs} := f \cdot \frac{c - v_o}{c + v_s} = f \cdot \left(1 - \frac{v_o + v_s}{c} - \frac{v_o \cdot v_s}{c^2} + \frac{{v_s}^2}{c^2} + \ldots \right)$$

whereas the relativistic form expands to

$$f_{obs} := f \cdot \left(1 - \frac{v}{c} + \frac{1}{2} \cdot \frac{v^2}{c^2} + \ldots \right)$$

Clearly the two forms agree to first order in $v/c = (v_s+v_o)/c$.

Each of the above expressions gives

$$\frac{f - f_{obs}}{f} := \frac{v}{c} - a \cdot \frac{v^2}{c^2} + O\left(\frac{v^3}{c^3} \right)$$

where a=1 in the non-relativistic formula, a=1/2 in the relativistic form. The relative error in the calculated refractive index will therefore also be of order (v/c), that is, one part in 10^6.