**MODEL ANSWERS AND MARKING NOTES**
**PHYS1B24 2001**

SECTION A **[Part marks]**

1. The *principle of superposition* states that when two waves exist in a medium the total disturbance is the algebraic sum of the disturbances of the two, so that **[1]**

$$\begin{aligned} y(x,t) &= A\ \mathrm{Re}\left[e^{i(\omega t-kx)}\right] + A\ \mathrm{Re}\left[e^{i(\omega t+kx)}\right] \\ &= A\ \mathrm{Re}\left[e^{i\omega t}\left(e^{-ikx}+e^{ikx}\right)\right] \\ &= 2A\ \mathrm{Re}\left[e^{i\omega t}\cos(kx)\right] \\ &= 2A\cos(\omega t)\cos(kx). \end{aligned}$$

**[3]**

$y_1$ and $y_2$ represent waves of equal amplitude travelling respectively to the right and to the left: their superposition represents a standing wave. *(Allow 1/2 mark for writing the total as a sum. Allow 1 mark for writing as the sum of two cosines.)* **[1,2]**

2. As the figure shows,

the path length difference between rays at an angle $\theta$ from successive slits a distance $d$ apart is $d\ \sin(\theta)$, and there will be constructive interference when this path difference is an integer number of wavelengths. *(1 mark will be deducted if there is no implicit or explicit mention of interference)* That is, **[4]**

$$d\sin(\theta) = n\lambda,$$

where $n$ is an integer.

If such a diffraction grating with 500 slits per mm is illuminated with 600 nm light, the maximum order of diffraction is the largest integer $n$ such that $\sin(\theta) <= 1$:

$$n <= d/\lambda = \frac{(10^{-3}/500)}{600\times10^{-9}} = 3.333,$$

i.e. the maximum order is 3. **[3]**

3. The change of phase which occurs when a light wave is reflected from (i) an optically more dense material is $\pi$, (ii) an optically less dense material is 0. *(Award 1 mark for $0$ and $\pi$ but interchanged)* **[2]**

The diagram shows the formation of interference rings in Newton's experiment *(must show interfering rays clearly — no marks for just drawing rings, as the question tells us to expect rings).* **[2]**

At a radius $r$, the thickness of the air layer $t$ is given by

$$r^2 = t(2R - t) \text{ the chord formula}$$

or

$$r^2 = R^2 - (R - t)^2 \text{ Pythagoras}$$

so, for small $t$, **[1]**

$$r^2 = 2Rt.$$

The different phase changes on reflection from the bottom of the lens and the top of the plate are equivalent to a half wavelength optical path, so there will be constructive interference if the path length difference $2t$ is equal to an odd number of half wavelengths **[1]** *(it is essential that this is explained)*. Thus bright rings occur when

$$r^2 = R(2n+1)(\lambda/2)$$

or **[1]**

$$r_n = \sqrt{(2n+1)\frac{\lambda}{2}R}.$$

4. For an incident wave in medium 1 with amplitude $A$ the intensity is $\frac{1}{2}Z_1A^2\omega^2$, giving rise to reflected and transmitted intensities $\frac{1}{2}Z_1r^2A^2\omega^2$ and $\frac{1}{2}Z_2t^2A^2\omega^2$ respectively. **[2]** Substituting the expressions given, the sum of the reflected and incident intensities is

$$\begin{aligned} & \frac{1}{2}Z_1\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 A^2\omega^2 + \frac{1}{2}Z_2\left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 A^2\omega^2 \\ = & \frac{1}{2}\left[\frac{Z_1(Z_1 - Z_2)^2 + 4Z_1^2Z_2}{(Z_1 + Z_2)^2}\right] A^2\omega^2 \\ = & \frac{1}{2}Z_1\left[\frac{(Z_1 + Z_2)^2}{(Z_1 + Z_2)^2}\right] A^2\omega^2 \end{aligned}$$

which is equal to the incident flux, showing that energy is conserved at the interface. **[3]**

If $Z_{\text{air}} = 400 \text{ kg m}^{-2} \text{ s}^{-1}$ and $Z_{\text{water}} = 1.45 \times 10^6 \text{ kg m}^{-2} \text{ s}^{-1}$ the amplitude transmitted to the water is **[2]**

$$10^{-6} \times \frac{2Z_{\text{air}}}{Z_{\text{air}} + Z_{\text{water}}} \text{ m} = 10^{-6} \times \frac{2 \times 400}{400 + 1.45 \times 10^6} \text{ m} = 5.5 \times 10^{-10}\text{m}$$

.

5. During each period $T$ of the source, it emits one wave but also moves towards the observer by a distance $v_sT = v_s/f$. The observed wavelength is decreased from $\lambda$ to $\lambda' = \lambda - v_s/f$. Thus the observed frequency $f'$ is *(note that half the marks are for the explanation, half for the formulae)* **[2]**

$$f' = \frac{c}{\lambda'} = \frac{c}{\lambda - v_s/f} = \frac{c}{c/f - v_s/f} = \frac{f}{1 - (v_s/c)}.$$

**[2]**

If the source is receding, the same formula applies with the sign of $v_{\mathrm{s}}$ reversed. Hence *(one mark for writing down the initial line giving the difference)* **[3]**

$$\Delta f = \frac{f}{1-(v_{\mathrm{s}}/c)} - \frac{f}{1+(v_{\mathrm{s}}/c)} = f\frac{1+(v_{\mathrm{s}}/c)-(1-(v_{\mathrm{s}}/c))}{(1-(v_{\mathrm{s}}/c))(1+(v_{\mathrm{s}}/c))} = \frac{2f(v_{\mathrm{s}}/c)}{1-(v_{\mathrm{s}}/c)^2}.$$

6. Ray diagram showing the formation of the image of a slide on a screen by a simple converging lens: **[3]**

Applying the lens formula *(allowing other sign conventions provided it is clear where the object is)*

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

gives

$$u = \frac{vf}{f-v} = \frac{5000 \times 50}{50 - 5000} = -50.5 \text{ mm},$$

that is, the slide is 50.5 mm from the lens. From the figure it is then obvious that the height of the image is **[2]**

$$h' = \frac{v}{u}h = \frac{5000}{-50.5}24 = -2376 \text{ mm},$$

or -2.4 m *(deduct no marks if sign not included here)*. **[2]**

7. Diagram of a Michelson interferometer: **[3]**

(Essential features: source, beam splitter, fixed and movable mirrors. Do not deduct marks if no compensating plate shown)

When the mirror moves through a distance $d$ the path length in that arm changes by $2d$. **[1]**
Each change of $\lambda$ in the path length shifts the fringe pattern by one dark fringe, so a shift of $n$ fringes corresponds to **[1]**

$$2d = n\lambda,$$

or **[2]**

$$d = \frac{100 \times 600 \times 10^{-9}}{2} = 3 \times 10^{-5} \text{ m}.$$

8. The required expressions are **[2]**

$$v_{\mathrm{p}} = \frac{\omega}{k}$$
$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

The energy in the wave propagates at the group velocity. **[1]**

The dispersion relation may be written **[2]**

$$\omega = \frac{a}{\lambda^2} = \frac{ak^2}{4\pi^2},$$

so

$$v_{\mathrm{p}} = \frac{ak}{4\pi^2}$$
$$v_{\mathrm{g}} = \frac{2ak}{4\pi^2},$$

and the group velocity is twice the phase velocity *(allow one mark for deriving* $v_{\mathrm{g}} = v_{\mathrm{p}} + k\partial v_{\mathrm{p}}/\partial k$*).* **[2]**

SECTION B

9. The figure shows the forces acting on a segment of the string between $x$ and $x + \delta x$, across which the transverse displacement varies from $y$ to $y + \delta y$.

The transverse force at $x$ is $T\sin(\theta) \approx T\left(\frac{\partial y}{\partial x}\right)_x$, acting in the negative $y$ direction, and that at $x+\delta x$ is $T\left(\frac{\partial y}{\partial x}\right)_{x+\delta x}$, acting in the positive $y$ direction. The mass of this segment is $\rho\delta x$, and so we may apply Newton's equation of motion to give **[3]**

$$\begin{aligned}\rho\delta x\frac{\partial^2 y}{\partial t^2} &= T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] \\ \rho\frac{\partial^2 y}{\partial t^2} &= T\frac{\partial^2 y}{\partial x^2},\end{aligned}$$

the wave equation with speed $v = \sqrt{T/\rho}$. **[4]**

The kinetic energy contained in a length $\delta x$ with a wave

$$y(x,t) = A\cos(kx - \omega t + \phi)$$

is

$$E_\mathrm{k} = \frac{1}{2}\rho\ \delta x\ A^2\ \omega^2 \sin^2(kx - \omega t + \phi)$$

and averaging over a wavelength gives **[3]**

$$\begin{aligned}\langle E_\mathrm{k}\rangle &= \frac{\int_0^\lambda \frac{1}{2}\rho\ A^2\ \omega^2 \sin^2(kx - \omega t + \phi)\mathrm{d}x}{\lambda} \\ &= \frac{\frac{1}{2}\rho\ A^2\ \omega^2 \int_0^\lambda \frac{1}{2}\left[1 - \cos(2kx - 2\omega t + 2\phi)\right]\mathrm{d}x}{\lambda} \\ &= \frac{1}{4}\rho\ A^2\ \omega^2\end{aligned}$$

This energy, and the equal amount stored as potential energy, propagate along the string with speed $v$, giving the total rate of energy transport along the string as $\frac{1}{2}v\ \rho\ A^2\ \omega^2$ *(2 marks are for the averaging, 1 for converting to rate of transport of energy. Allow 1 mark for quoting $\frac{1}{2}mv^2$, 1 for a reasonably clear reference to averaging over a wavelength or period).* **[3]**

If waves of amplitude 10 cm are to be transmitted along a string of linear density $10\ \mathrm{g\ m}^{-1}$ which is under a tension of 100 N, the highest frequency achievable with a source power of 1 kW is $f_\mathrm{max}$ given by **[2]**

$$10^3 = \frac{1}{2}\sqrt{\frac{100}{10 \times 10^{-3}}}\ 10^{-2}\ 0.1^2\ (2\pi)^2 f_{\max}^2$$

so that **[1]**

$$f_{\max} = 71.2 \text{ Hz}.$$

The lowest frequency standing wave has a wavelength equal to twice the length $L$ of the string and thus its frequency is $v/\lambda = v/2L$. The required length is then **[3]**

**[1]**

$$L = v/2f = \frac{1}{2}\sqrt{\frac{100}{10 \times 10^{-3}}}/71.2 = 0.7 \text{ m}.$$

10. The diagram shows rays passing through a Fabry-Perot interferometer

showing that interference is between rays which pass through the etalon at the same angle, and emphasising the importance of silvering or dielectric coating on the inner surfaces of the etalon. **[4]**

If the amplitude of transmission of a ray through the first plate of the etalon is $t$, that through the second $t'$, the reflection coefficients from the inside of the first and second plates are $r$ and $r'$, and the phase change involved in a double traverse of the gap and the two reflections is $\delta$, then the amplitude transmitted through the etalon may be written as **[2]**

**[2]**

$$E_t = E_0\left[tt' + tr'rt'e^{i\delta} + tr'rr'rt'e^{2i\delta} + ....\right]$$

(note that any change in phase involved in the straight-through passage has been incorporated with $E_0$, so that $t$, $t'$, $r$ and $r'$ are real). This is a geometric progression with first term $E_0tt'$ and common ratio $r'r\exp(i\delta)$. Summing the series to infinity gives **[3]**

$$E_t = E_0\left[\frac{tt'}{1 - rr'e^{i\delta}}\right].$$

*(Allow 2 marks for a clear explanation of all the symbols in the result without a derivation)*

To determine the intensity we compute $E_tE_t^*$: **[1]**

$$\begin{aligned} I &\propto \frac{(tt')^2}{1 - 2rr'\cos(\delta) + (rr')^2} \\ &= \frac{(tt')^2}{1 - 2rr' + 4rr'\sin^2(\delta/2) + (rr')^2} \\ &= \frac{(tt')^2/(1-rr')^2}{1 + [4rr'/(1-rr')^2]\sin^2(\delta/2)} \end{aligned}$$ **[2]**

Introducing

$$F = \frac{4rr'}{(1-rr')^2},$$

we may write **[1]**

$$I \propto \frac{(tt'/rr')^2 F}{1 + F\sin^2(\delta/2)}).$$

Clearly $I$ peaks when $\sin(\delta/2) = 0$, so **[1]**

$$I = \frac{I_0}{1 + F\sin^2(\delta/2)}.$$

The figure below shows how the intensity pattern varies with $F$, and shows that large $F$, corresponding to $r$ and $r'$ close to 1, increases the sharpness of the pattern – hence the need for coatings on the inner surfaces of the etalon. **[4]**

11. The diagram shows Young's experiment for generating interference fringes from a spatially incoherent source: it is essential that the first slit be included — deduct 1 mark if not, or if source not described as coherent. **[3]**

If the apparatus is set up with two identical slits with centres $d$ apart, the amplitude at a screen a distance $x$ from the screen with light of wavelength $\lambda = 2\pi/k$ will be *(2 marks for superposition, 1 for distance formula)* **[3]**

$$E(y,t) = E_0 e^{-i\omega t}\left[e^{ik\sqrt{x^2+(y-d/2)^2}} + e^{ik\sqrt{x^2+(y+d/2)^2}}\right].$$

We now make the approximation that $y,\ d << x$, so **[2]**

$$\begin{aligned}\sqrt{x^2+(y\pm d/2)^2} &= x\left[1+\left(\frac{y\pm d/2}{x}\right)^2\right]^{1/2} \\ &\approx x\left[1+\left(\frac{y^2+(d/2)^2}{2x^2}\right)\pm\frac{yd}{2x^2}\right]\end{aligned}$$

Writing

$$D = x\left[1+\frac{y^2+(d/2)^2}{2x^2}\right]$$

we have **[1]**

$$\begin{aligned}E(y,t) &= E_0 e^{i(kD-\omega t)}\left[e^{-ikdy/2x}+e^{ikdy/2x}\right] \\ &= E_0 e^{i(kD-\omega t)}\cos(kdy/2x).\end{aligned}$$

Thus **[1]**

$$I(y) = I_0\cos^2(\pi dy/\lambda x).$$

*(Full marks for clear derivation using plane waves at angle $\theta$)*

The maximum amplitude occurs when the signals from the two slits add in phase, the minimum when they are exactly out of phase. Thus for amplitudes $A$ and $B$ we have **[2]** $I_{\max} \propto (A+B)^2$ and $I_{\min} \propto (A-B)^2$. Hence the visibility of the fringe pattern is *(1 mark for defining $V$, 3 for the algebra)* **[4]**

$$\begin{aligned}V &= \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \\ &= \frac{(A+B)^2 - (A-B)^2}{(A+B)^2 + (A-B)^2} \\ &= \frac{4AB}{2\left(A^2+B^2\right)^2} \\ &= \frac{2A/B}{1+(A/B)^2.}\end{aligned}$$

A slip of glass $t$ thick with refractive index 1.5 introduces a change in optical path length of $(1.5-1)t$ (we are looking near the centre of the pattern, so there are no angular corrections). If this does not alter the pattern, this optical path length must be an integral number of wavelengths. Thus

$$(1.5-1)t = p\lambda,$$

where $\lambda$ is the wavelength and $p$ is an integer, so

$$t = p \times 500 \times 10^{-9}/0.5 = p \times 10^{-6}\ \mathrm{m},$$

i.e. an integral number of microns. **[6]**

12. The cases expected are spherical and chromatic aberration, with the use of a doublet to reduce the effects of chromatic aberration, as shown in the diagram below, but note that the question (deliberately) did not specify spherical lenses so that barrel and similar distortions are also acceptable. *(For correcting chromatic aberration, mention of higher dispersion of correcting element is essential)* **[6]**

The diagram shows a compound microscope with an objective with focal length 4 mm and an eyepiece with focal length 20 mm, forming an image, 250 mm from the eye, of an object placed 4.1 mm from the objective. *(The diagram must show the intermediate and final images, which must be located using two or more principal rays for each image)* **[5]**

The image at 250 mm gives us, from the lens formula for the eyepiece *(other sign conventions may be used, but must be correctly applied)* **[1]**

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

the position of the intermediate image as

$$\frac{1}{u} = \frac{1}{-250} - \frac{1}{20}$$

or

$$u = -500/27\ \mathrm{mm} = -18.5\ \mathrm{mm}.$$

This point, 18.5 mm to the left of the eyepiece, is $L - 18.5$ mm to the right of the **[1]** objective. But for the objective [1]

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u} = \frac{1}{4} + \frac{1}{-4.1} = \frac{0.1}{16.4},$$

or $v = 164$ mm. Thus the separation of the lenses in this setting is $18.5 + 164 =$ [1]
$182.5\,\mathrm{mm}$. The magnification of the microscope when adjusted in this way is the product [1]
of the objective and eyepiece magnifications [2]

$$M = \frac{164}{-4.1} \times \frac{250}{18.5} = -540.$$

(The sign denotes that the image is inverted, but deduct no marks if the sign is omitted)
It is more usual to adjust the microscope so that the image is at infinity because the eye [2]
is then relaxed, and so use of the microscope is less tiring.

13. The key difference between Fraunhofer and Fresnel diffraction is that in the former case the curvature of the wavefronts may be neglected, but in the latter it may not (accept large distance between diffracting element and detection point for Fraunhofer limit). [2]

The field distribution in the Fraunhofer pattern of a slit of width $w$ illuminated with light of wavelength $\lambda = 2\pi/k$ may be computed by considering a wave leaving the slit at an angle $\theta$, when the phase difference between the point $y$ on the slit and the centre of the slit (see diagram) is [2]

$$\phi(y) = ky\sin(\theta)$$

and the total field is thus [3]

$$\begin{aligned} E(\theta) &= E_0 \int_{-w/2}^{w/2} e^{-iky\sin(\theta)}\mathrm{d}y \\ &= E_0 \left[\frac{e^{-iky\sin(\theta)}}{-ik\sin(\theta)}\right]_{-w/2}^{w/2} \\ &= E_0 \frac{2\sin\left(\frac{kw\sin(\theta)}{2}\right)}{k\sin(\theta)} \end{aligned}$$

and the limiting value as $\theta \to 0$ is $E(0) = wE_0$, so [1]

$$E(\theta) = E(0)\frac{\sin\left(\frac{kw\sin(\theta)}{2}\right)}{kw\sin(\theta)/2}$$

and, defining

$$\beta = kw\sin(\theta)/2 = \frac{\pi w}{\lambda}\sin(\theta),$$

we have [1]

$$I(\theta) = I(0)\left[\frac{\sin(\beta)}{\beta}\right]^2.$$

*(Allow the alternative approach to the slit as the limiting case of a diffraction grating with $N$ slits each of width $h$, but limit marks to 5 if limits $N \to \inf$, $h \to 0$, $Nh \to w$ not taken correctly)*

The figure illustrates Rayleigh's criterion for the resolution of images formed by a slit, that the images can be resolved if the central maximum of one diffraction pattern falls on or beyond the first minimum of the other. The expression just derived shows that **[2]** the first minimum occurs when $\beta = \pi$, or $\sin(\theta) = \lambda/w$. If we assume that the slit is large compared with the wavelength, $\sin(\theta) \approx \theta$ and we obtain $\theta \geq \lambda/w$. For a circular **[3]** aperture of diameter $D$ the corresponding result is $\theta \geq 1.22\lambda/D$. **[2]**

Thus a human eye with a pupil diameter of 2.5 mm observing at a wavelength of 500 nm can resolve points with angular separation $\theta = 1.22 \times 500 \times 10^{-9}/2.5 \times 10^{-3} = 2.44 \times 10^{-4}\,\mathrm{radians}$. Thus at a range of 250 mm the minimum separation is $250 \times 2.44 \times 10^{-4} = 6.1 \times 10^{-2}\,\mathrm{mm}$ (for interest, about a hair's breadth). **[4]**

PHYS1B24/2001 **END OF SOLUTIONS**