Part B

9. Consider the equation of motion of an element of fluid with cross-sectional area A perpendicular to the x axis, of infinitesimal thickness dx, and let the displacement of particles at the point x at time t be $\xi(x,t)$.

The mass of this element will be $\rho \cdot A \cdot dx$.

According to the definition of the bulk modulus, the excess pressure at x is the negative of the product of the bulk modulus and the volume strain at x. The volume strain is the change in volume per unit volume, that is, $(\xi(x+dx,t) - \xi(x,t))/dx$, which, by the basic definition of the derivative, is $d\xi(x,t)/dx$. Thus the pressure on the element at x is thus $- A\ B\ d\xi(x,t)/dx$.

Similarly, the force on the other face of the element, at $x+dx$, is $- A\ B\ d\xi(x+dx,t)/dx$.

We may now write down Newton's equation of motion for the element:

$$\rho\ A\ dx\ (d^2\xi(x,t)/dt^2) = A\ B\ (d\xi(x+dx,t)/dx - d\xi(x,t)/dx\ )$$

and divide through by $\rho\ A\ dx$, recognise the appearance of the second derivative on the right-hand side, to obtain

$$d^2\xi(x,t)/dt^2 = (B/\rho)\ d^2\xi(x,t)/dx^2$$

that is, a wave equation with velocity $\sqrt{\dfrac{B}{\rho}}$.

The specific acoustic impedance is the ratio of the excess pressure to the particle velocity. Consider a cosinusoidal wave $\xi(x,t) := a \cdot cos(k \cdot x - \omega \cdot t)$. The excess pressure is

$$p(x,t) := a \cdot B \cdot k \cdot sin(k \cdot x - \omega \cdot t)$$

and the particle velocity is

$$v(x,t) := a \cdot \omega \cdot sin(k \cdot x - \omega \cdot t)$$

so the specific impedance is

$$Z = B \cdot \left(\frac{k}{\omega}\right) = \sqrt{B \cdot \rho}.$$

In the reflection at an interface, conservation of energy demands that the rate of energy input (power) in the incident wave should equal the sum of the powers in the reflected and transmitted waves. The power per area is the product of the energy per volume and the wave speed. Thus, if the amplitude transmission coefficient is t_{12} we have (using an obvious notation for the velocities in the different media)

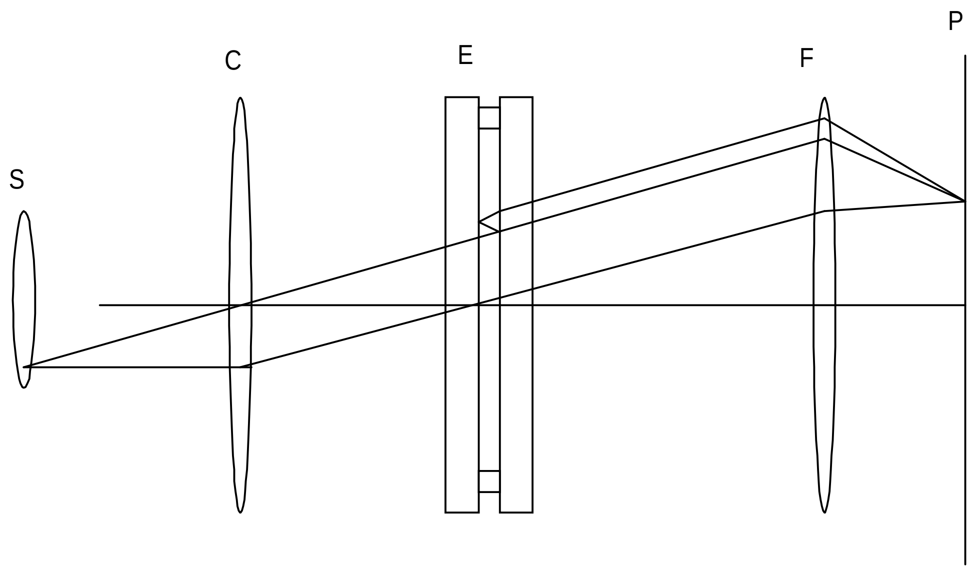
$$\frac{1}{2} \cdot \rho_1 \cdot \omega^2 \cdot v_1 = \frac{1}{2} \cdot \rho_1 \cdot \omega^2 \cdot v_1 \cdot \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 + \frac{1}{2} \cdot \rho_2 \cdot \omega^2 \cdot v_2 \cdot \left(t_{12}\right)^2$$

noting that $v = \dfrac{Z}{\rho}$, we have

$$Z_1 = Z_1 \cdot \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 + Z_2 \cdot \left(t_{12}\right)^2 \quad \text{whence} \quad \left| t_{12} \right| = \frac{2 \cdot Z_1}{Z_1 + Z_2}.$$

For air the speed of sound is $\sqrt{\dfrac{\frac{7}{5} \cdot 0.1 \cdot 10^6}{1.28}} = 331$ m/s, and for Helium we have $\sqrt{\dfrac{\frac{5}{3} \cdot 0.1 \cdot 10^6}{0.18}} = 962$ m/s. The pitch of the voice is determined by the resonant frequencies of the vocal tract, so if the dimensions are fixed but the speed of sound increases the frequency will also increase.

10. The diagram shows the Fabry-Perot interferometer, with key components marked.



S source (which may be extended)
C collimating lens (which may be omitted with a very wide source)
E etalon: note that a) the inner surfaces are coated to increase reflectivity
b) the outer surfaces may be angled slightly to avoid reflections
The diagram illustrates one internal reflection only.
F focussing lens
P screen in focal plane of F.

At normal incidence, let the transmission coefficient from glass to air at the left-hand side of the etalonn air gap be t_{12}, that from air to glass on the other side be t_{23}. Also let the reflection coefficient in air at the left-hand side of the gap be r_{21}, that at the right-hand side be r_{23}. On passing once across the air gap the phase of the light changes by $k\,d$, where k is the wavevector and d is the thickness of the air gap.

Then, if the field in the glass at the right-hand side of the etalon gap from the ray which travels through without reflection is $t_{12}\cdot t_{23}\cdot E_0$, the total field in the glass just beyond the air gap will be the sum of

this straight-through signal $t_{12}\cdot t_{23}\cdot E_0$

the signal which has been internally reflected twice $t_{12}\cdot r_{23}\cdot e^{2\cdot i\cdot k\cdot d}\cdot r_{21}\cdot t_{23}$

the signal which has been internally reflected four times, $t_{12}\cdot\left(r_{23}\cdot e^{2\cdot i\cdot k\cdot d}\cdot r_{21}\right)\cdot\left(r_{23}\cdot e^{2\cdot i\cdot k\cdot d}\cdot r_{21}\right)\cdot t_{23}$

where k is the wavevector.

This is clearly a geometric series, which we simplify by writing $T = t_{12}\cdot t_{23}$ and $R = r_{23}\cdot r_{21}$ and sum to infinity to find the total field

$$E = E_0\cdot T\cdot\frac{1}{1 - R\cdot e^{2\cdot i\cdot k\cdot d}}$$

and hence the intensity may be obtained from the square modulus of E.

$$I = I_0\cdot T^2\cdot\frac{1}{\left(1 + R^2\right) - 2\cdot R\cdot cos(2\cdot k\cdot d)}$$

Now use $cos(2\cdot\theta) = 1 - 2\cdot sin(\theta)^2$

$$I = I_0 \cdot T^2 \cdot \frac{1}{\left(1 + R^2 - 2 \cdot R\right) + 4 \cdot R \cdot sin(k \cdot d)^2}$$

Finally

$$I = I_0 \cdot \left(\frac{T}{1 - R}\right)^2 \cdot \frac{1}{1 + F \cdot sin\left(2 \cdot \pi \cdot \frac{d}{\lambda}\right)^2}$$

where

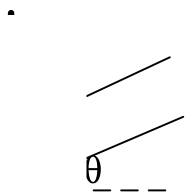
$$F = \frac{4 \cdot R}{(1 - R)^2}$$

The intensity is a maximum when the sin term is zero, maximum when it is unity, so the ratio of maximum to minimum transmitted intensity is $(1+F)$.

Thus if F=0.5 the ratio is 1.5.

If F=500 the ratio is 501.

11. The coherence of a wave pattern describes the extent to which it can be described as a continuous wave with a fixed frequency with a smooth wavefront. Longitudinal coherence describes the extent to which the spatial variation along the direction of travel (or, equivalently, the variation in time at a particular point) resembles an infinite sinusoid. Transverse coherence describes the distance perpendicular to the direction of travel over which the phase of the wave remains constant.



The difference in path length (see diagram) between successive slits in the Fraunhofer regime is $d\cdot sin(\theta)$. Thus the total field seen at an angle θ may be written in complex form as

$$E = E_1 + E_1\cdot e^{i\cdot k\cdot d\cdot sin(\theta)} + E_1\cdot e^{2\cdot i\cdot k\cdot d\cdot sin(\theta)} \ldots$$
$$+ E_1\cdot e^{(N-1)\cdot i\cdot k\cdot d\cdot sin(\theta)}$$

where E_1 is the field from slit number 1. Summing this geometric progression gives

$$E = E_1\cdot\frac{1-e^{N\cdot i\cdot k\cdot d\cdot sin(\theta)}}{1-e^{i\cdot k\cdot d\cdot sin(\theta)}} = E_1\cdot\frac{e^{\frac{N}{2}\cdot i\cdot k\cdot d\cdot sin(\theta)}}{e^{\frac{1}{2}\cdot i\cdot k\cdot d\cdot sin(\theta)}}\cdot\frac{sin\left(\frac{N}{2}\cdot k\cdot d\cdot sin(\theta)\right)}{sin\left(\frac{1}{2}\cdot k\cdot d\cdot sin(\theta)\right)}.$$

Now form the intensity from the square modulus of *E*, noting that for narrow slits the field of each slit is cylindrically symmetrical, independent of θ

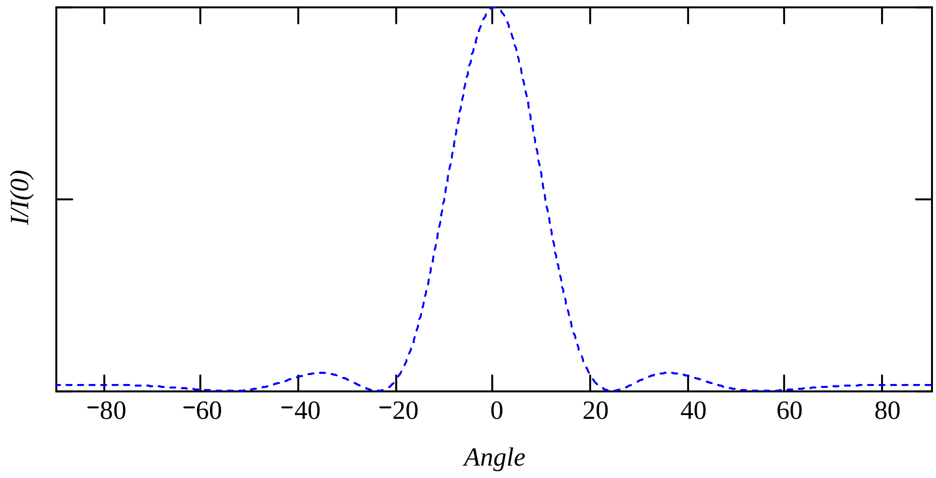
$$I = I_1\cdot\left(\frac{sin(N\cdot\gamma)}{sin(\gamma)}\right)^2 .$$

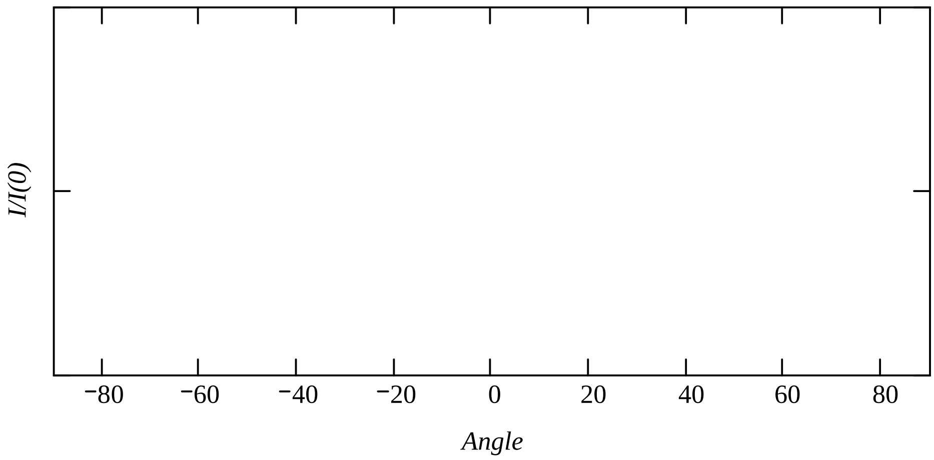
Finally, note that at θ=0 the ratio of the sines is equal to *N*, either by l'Hopital's rule or from the physical argument that there all the fields add in phase,so that

$$I = I(0)\cdot\left(\frac{sin(N\cdot\gamma)}{N\cdot sin(\gamma)}\right)^2 ,$$

where $\gamma = \frac{1}{2}\cdot k\cdot d\cdot sin(\theta) = \frac{\pi\cdot d\cdot sin(\theta)}{\lambda}$.

In the sketch below, the dotted line shows the finite slit pattern, which will alter the diffraction pattern as in the next figure.





The maxima of the diffraction pattern occur when γ is an integral multiple of π, that is for

$\theta_n = \arcsin\left(\frac{n\cdot\lambda}{d}\right)$. The maximum order is the largest integer less than $\frac{d}{\lambda}$, $\frac{\left(\frac{10^{-3}}{600}\right)}{600\cdot 10^{-9}} = 2.778$, i.e. 2.

Here we have $\operatorname{asin}\left(\frac{1\cdot 600\cdot 10^{-9}}{\frac{10^{-3}}{600}}\right) = 0.368$ radians, $\operatorname{asin}\left(\frac{2\cdot 600\cdot 10^{-9}}{\frac{10^{-3}}{600}}\right) = 0.804$ radians,

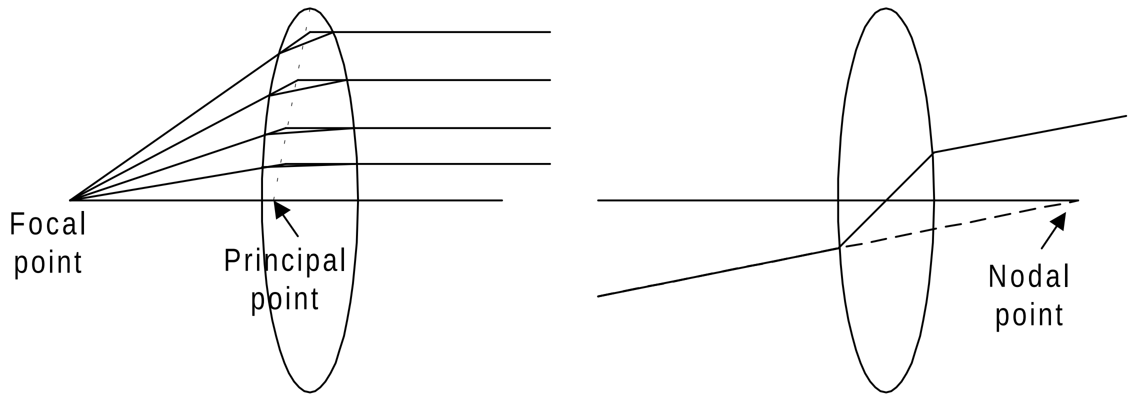
in addition to the straight-through signal at 0 (21.1 and 46.1 degrees).

12. The *principal points* of the lens system are the points at which the principal planes intersect the axis. The principal plane is the planar approximation to the surface formed by the locus of the points of intersection of the rays parallel to the axis and the corresponding rays refracted through the principal foci.

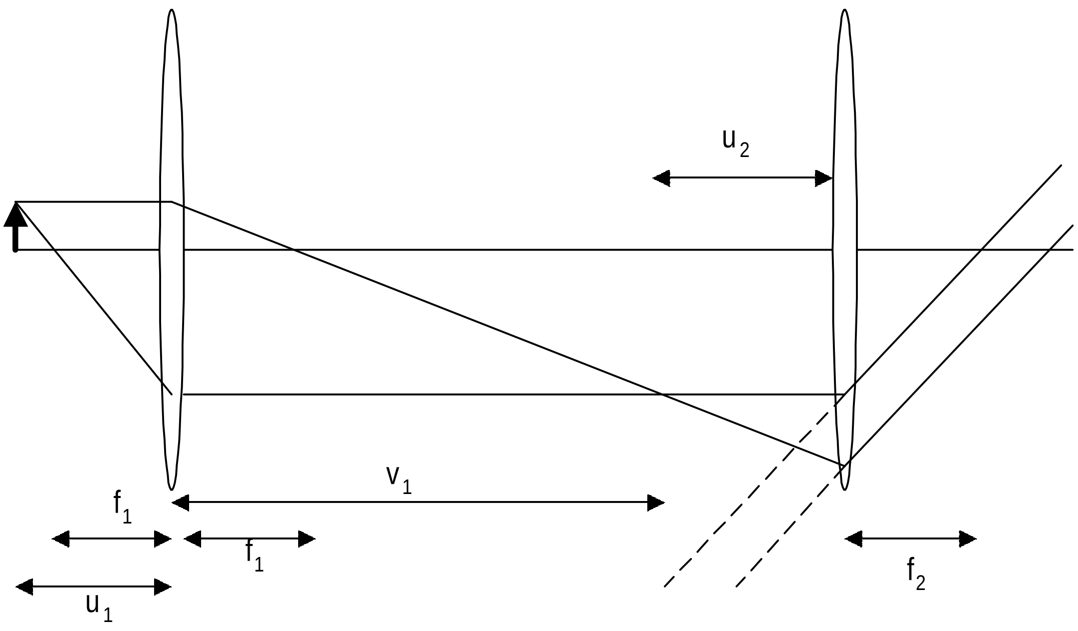
The *principal foci* of the lens system are the points at which a ray (or its projection) incident parallel to the axis of the lens system crosses the axis after passing through the system.

The *nodal points* are the points of intersection with the axis of rays which pass through the optical centre of the system, that is, rays which pass through the lens without being deviated in angle.

In the diagrams below, only one set of the foci and principal cardinal points is shown: a corresponding diagram for rays from the other side (which need not be given) will show the other set.



The compound microscope, set up with the image at infinity.



Apply the lens formula separately to the two lenses. The image in the second lens is at infinity, so its object must be a distance f_2 to the left of lens 2. As the lenses are 220 mm apart, and the focal length of lens 2 is 20 mm, this places the object of lens 2, which is the image from lens 1, 200 mm to the right of lens 1. For lens 1, then,

$$\frac{1}{200} - \frac{1}{u_1} = \frac{1}{f_1} = \frac{1}{5}$$

or $u_1 = -\frac{1000}{195} = -5.128$

which places the object 5.1 mm to the left of the objective.

The magnifying power of the microscope combines the linear magnification of the objective,

$$M_1 = \frac{v_1}{u_1}$$

with the angular magnification of the eyepiece

$$M_2 = \frac{250}{f_2}$$

giving a total magnifying power of

$$\frac{200}{-5.128} \cdot \frac{250}{20} = -487.52$$

or approximately 490 magnification, with inverted image.

Note that the correct result may also be obtained by using *M=(near point distance)/f* with *f* equal to the focal length
of the composite lens, calculated from

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 \cdot f_2}$$

and that this is entirely valid in this case.

13. In Fraunhofer diffraction, both the incident wave and the diffracted wave may be assumed to be plane. This means that the incident wave is either from a distant source or the light has been collimated before reaching the diffracting screen, or it has been collimated by a lens. In Fresnel diffraction, curvature of the wavefront is taken into account.

For a slit of width w the difference in path length between rays from the centre and the edge of the slit at a distance D from the aperture, y away from the centre, is

$$\Delta = \sqrt{D^2 + \left(y + \frac{w}{2}\right)^2} - \sqrt{D^2 + y^2}$$

which, for large D, we may expand as

$$\Delta = \frac{1}{2 \cdot D}\left(y + \frac{w}{2}\right)^2 - \frac{1}{2 \cdot D} \cdot y^2 = \frac{w \cdot y}{2 \cdot D} + \frac{w^2}{8 \cdot D}.$$

Now we can neglect the change in phase associated with the second term (involving curvature) if it amounts to less than one-eighth of a wavelength. The Rayleigh distance is thus defined by

$$\frac{\lambda}{8} > \frac{w^2}{8 \cdot D} \quad \text{or} \quad D > \frac{w^2}{\lambda}.$$

In the Fraunhofer limit, the path difference between a ray passing through the centre of the aperture at an angle θ to the normal and another ray at the same angle a distance x from the centre is x sin(θ), which
corresponds to a phase difference $k\,x$ sin(θ) for light with a wavevector k.

If the contribution to the final amplitude from an element of the slit of width is E_0 dx, the total field in complex exponential notation may be written as

$$E = E_0 \cdot \int_{\frac{-w}{2}}^{\frac{w}{2}} e^{i \cdot k \cdot x \cdot \sin(\theta)}\, dx = \frac{1}{i \cdot k \cdot \sin(\theta)} \cdot \left(e^{i \cdot k \cdot \frac{w}{2} \cdot \sin(\theta)} - e^{-i \cdot k \cdot \frac{w}{2} \cdot \sin(\theta)}\right)$$

Rewriting the difference of the complex exponentials as 2i times the sine, we have

$$E = E_0 \cdot 2 \cdot \frac{\sin\left(k \cdot \frac{w}{2} \cdot \sin(\theta)\right)}{k \cdot \sin(\theta)}$$

Now define $\beta = k \cdot \frac{w}{2} \cdot \sin(\theta)$, convert to intensity, and use the limiting form of the sine as θ tends to zero to write

$$I(\theta) = I_0 \cdot \left(\frac{\sin(\beta)}{\beta}\right)^2,$$

where I_0 is the intensity at θ=0.

The first zero of this pattern comes when $k \cdot \frac{w}{2} \cdot \sin(\theta) = \pi$, or, for small θ and writing 2π/λ for k,

θ = λ/w. Rayleigh's condition for resolution is that the central maximum of one source's diffraction pattern must lie over or beyond the first minimum of the other, so that the slits can be resolved if their angular separation exceeds λ/ w.

For a circular aperture, the condition is modified to 1.22 λ/ w, where w is the diameter of the aperture.

The angular resolution of the HST at 120 nm, then, is $1.22 \cdot \frac{120 \cdot 10^{-9}}{2.4 \cdot 2} = 3.05 \cdot 10^{-8}$ radians or

$1.22 \cdot \frac{120 \cdot 10^{-9}}{2.4 \cdot 2} \cdot \frac{180 \cdot 60 \cdot 60}{\pi} = 6.291 \cdot 10^{-3}$ seconds of arc.