QUEEN MARY,
UNIVERSITY OF LONDON
M.Sc. Astrophysics
ASTM003 Angular Momentum and Accretion in Astrophysics

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The duration of this examination is one and a half hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination.

The following notation is used throughout unless otherwise stated: The pressure, density, surface density, and temperature are denoted by P, ρ, Σ, and T respectively. The effective temperature is denoted by T_{eff}, and opacity by κ. The mean molecular weight, gas constant, and kinematic viscosity are denoted by μ, $\mathcal{R}$, and ν, respectively.

The gravitational constant $G = 7 \times 10^{-11}$ Kg^{-1} m^3 s^{-2}. Stefan's constant $\sigma = 6 \times 10^{-8}$ J m^{-2} s^{-1} K^{-4}. The speed of light $c = 3 \times 10^8$ m s^{-1}. The solar mass $M_\odot = 2 \times 10^{30}$ Kg. The solar radius $R_\odot = 7 \times 10^8$ m. Electron scattering opacity $\kappa = 0.04$ m^2 Kg^{-1}. The gas constant $\mathcal{R} = 8 \times 10^3$ J K^{-1} Kg^{-1}.

The following mathematical identities may be useful:

$\nabla.\mathbf{A}$ in cylindrical polar coordinates:

$$\nabla.\mathbf{A} = \frac{1}{R}\frac{\partial (RA_R)}{\partial R} + \frac{1}{R}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Trigonometric Identities:

$$\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

SECTION A *You should attempt all questions. Marks awarded are shown next to the questions.*

1) [*15 marks*] Accretion of matter onto neutron stars and black holes may occur at a maximum rate given by the Eddington limited accretion rate. Explain briefly the physical origin of this upper limit for the accretion rate.

Derive the following expression for the Eddington limited accretion rate, $\dot{m}_{Edd}$, onto a compact object of mass M:

$$\dot{m}_{Edd} = \frac{40\pi GM}{c\kappa} \tag{1}$$

Here κ is the opacity of the accreting gas, c is the speed of light, G is the gravitational constant. When deriving equation (1) you may use the following facts and assumptions:

The luminosity L is related to the radiative flux F by $L = 4\pi r^2 F$ ($r \equiv$ radius).

The radiative flux is given by the expression:

$$F = -\frac{c}{\kappa\rho}\frac{d}{dr}\left(\frac{aT^4}{3}\right)$$

The radiation pressure is given by $P_{rad} = aT^4/3$, where a is the radiation constant.

Luminosity generated by mass accretion at a rate $\dot{m}$ onto an object of mass M and radius R is given by $L = GM\dot{m}/R$.

You should assume that the radius of the accreting object $R = 5R_S$ where R_S is the Schwarzschild radius.

Estimate the Eddington limited accretion rate onto a 10^8 solar mass black hole, assuming that the opacity is produced by electron scattering. Estimate the effective temperature T_{eff} at a radius of $R = 5R_S$ in an accretion disc orbiting the black hole through which the matter is accreting. Comment on the temperature obtained in relation to the observation that X-rays are emitted by active galactic nuclei.

2)[*15 marks*] The virial theorem for an isolated fluid mass (e.g. a molecular cloud) acting under the forces of self–gravity and pressure may be written

$$\frac{d^2 I}{dt^2} = 4\mathcal{K} + 6\int_V P dV + 2E_g$$

where I is the moment of inertia, $\mathcal{K}$ is the total kinetic energy of the fluid mass, P is the pressure, and E_g is the total gravitational potential energy given by

$$E_g = -\frac{1}{2}G\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV dV'$$

where $\rho(\mathbf{r})$ is the density at position $\mathbf{r}$ in the cloud.

Consider a spherical isothermal molecular cloud of mass M, rotating uniformly with angular velocity Ω, and with radius R. The cloud has internal turbulent motions such that the magnitude of the maximum turbulent velocity is $|v_{max}|$. By assuming that the constituent gas obeys the ideal gas law, show that a sufficient condition for gravitational collapse of this cloud may be written as

$$\frac{GM}{2R} > 2R^2\Omega^2 + 2|v_{max}|^2 + \frac{6\mathcal{R}T}{\mu}.$$

Consider a cloud with temperature $T = 10$ K, internal turbulent motions with $|v_{max}| = 10$ m s^{-1}, rotational angular velocity $\Omega = 10^{-14}$ rad s^{-1}, radius $R = 3 \times 10^{15}$ m, and with mean molecular weight $\mu = 2$. Estimate the minimum mass that such a cloud requires in order to collapse to form a star.

3) [*20 marks*] Consider a close binary system composed of two stars of mass m_1 and m_2 in circular orbit about their common centre of mass. Working in a rotating Cartesian coordinate system that corotates with the orbital motion of the binary, and whose origin lies at the centre of mass, the coordinates of the two stars in the orbital plane are $(x_1, y_1) = (x_1, 0)$ and $(x_2, y_2) = (x_2, 0)$. The separation between the two stars is given by $D = x_2 - x_1$. Show that the total (gravitational plus centrifugal) potential at an arbitrary point (x, y) is given by:

$$\Phi(x,y) = -\frac{Gm_1}{\sqrt{(x-x_1)^2+y^2}} - \frac{Gm_2}{\sqrt{(x-x_2)^2+y^2}} + \frac{1}{2}\Omega^2(x^2+y^2) \tag{2}$$

where Ω is the angular velocity of the rotating reference frame.

The L_1 Lagrange point is a point of equilibrium in this system. Show that the following expression holds at the L_1 point, where x_L represents the x–coordinate of the L_1 point:

$$m_1\frac{(x_L-x_1)}{|x_L-x_1|^3} + m_2\frac{(x_L-x_2)}{|x_L-x_2|^3} + \frac{(m_1+m_2)}{D^3}x_L = 0 \tag{3}$$

Draw a sketch of the equipotentials produced by equation (2) for a binary system with $m_1 = m_2$. On your diagram you should indicate the behaviour of the forces in the near-vicinity of the individual stars and at large distance from the stellar components, the Roche lobes, and the position of the L_1 point.

For a binary system with $m_1 = 2$ solar masses, $m_2 = 0.5$ solar masses, and $D = 10^9$ m, equation (3) is satisfied for $x_L = 5 \times 10^8$ m (assuming that star m_1 lies on the negative x-axis and star m_2 lies on the positive x-axis). Show that mass transfer from star m_2 into the Roche lobe of star m_1 would result initially in the formation of a gaseous ring of approximate radius $R = 3 \times 10^8$ m, in orbit around star m_1. Comment on the significance of this result given the fact that a two solar mass main sequence star has a radius that is greater than the solar radius.

The viscosity operating in this gaseous ring is characterised by a value of the dimensionless viscosity coefficient $\alpha = 0.1$, and the ratio of height to radius of the ring is given by $H/R = 0.01$. Estimate the time required for the ring to spread due to the action of viscosity so that mass begins to accrete onto the star m_1. You should assume that the radius of the star m_1 is much less than the initial radius of the ring.

SECTION B *Each question carries 50 marks. You may attempt all questions but only marks for the best question will be counted.*

1) [*50 marks*] Consider an axisymmetric accretion disc with surface density Σ, kinematic viscosity ν, and in which forces due to pressure and self–gravity may be neglected. Derive the disc surface density evolution equation in the form

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial (R^2\Omega)}{\partial R}\right)^{-1}\frac{\partial}{\partial R}\left(R^3\nu\Sigma\frac{d\Omega}{dR}\right)\right] = 0$$

where R is the radial coordinate and Ω is the angular velocity.

You may assume that the torque acting in the direction of speeding the disc up, due to material interior to R, is given by

$$\mathcal{T} = -2\pi R^3\nu\Sigma\frac{d\Omega}{dr}.$$

Show that if the disc is in a state of Keplerian rotation, the surface density evolution equation is

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(R^{1/2}\nu\Sigma\right)\right].$$

In a steady state disc

$$\nu\Sigma = \frac{\dot{m}_d}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

where $\dot{m}_d$ is the constant rate of inwards mass flow in the disc and R_* denotes the position in the disc where $d\Omega/dR = 0$. The rate of energy production per unit area due to viscous dissipation in an accretion disc is

$$\epsilon_D = R^2\nu\Sigma\left(\frac{d\Omega}{dR}\right)^2$$

Show that the effective temperature profile in a steady state accretion disc is given by

$$T_{eff}^4 = \frac{3GM\dot{m}_d}{8\pi R^3\sigma}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

Show that only half of the available energy is radiated by accreting matter from infinity to radius R_* in a steady accretion disc, and explain how the remaining energy is stored. Describe a mechanism that may lead to the liberation of the remaining available energy.

2) [*50 marks*] Consider a protoplanet of mass m_p on a circular orbit in an axisymmetric Keplerian disc at radius R. The disc material interacts with the protoplanet as it shears past it, leading to angular momentum exchange. Consider only those parts of the disc that lie exterior to the protoplanet position. The interaction between disc and protoplanet can be described using the local impulse approximation. Here a Cartesian coordinate system is used that is centred on, and corotates with, the protoplanet.

The unperturbed disc flow past the the protoplanet occurs in the y direction only, with

$$v_y = u = -R\frac{d\Omega}{dR}x,$$

where Ω is the disc angular velocity at the protoplanet position. The equation of motion governing the deflection experienced by disc material due to an encounter with the protoplanet is

$$\frac{dv_x}{dt} = -\frac{Gm_p a}{(a^2+y^2)^{3/2}}$$

where v_x is the x component of the velocity and a is the impact parameter of the unperturbed disc material with respect to the protoplanet. Show that the value of v_x after the encounter is given by

$$v_x^2 = 4\left(\frac{Gm_p}{ua}\right)^2.$$

The angular momentum exchanged per unit mass during an encounter with the protoplanet is $R\Delta v_y$, where Δv_y is the change in the y component of the velocity. The rate of change of angular momentum of disc material is the change in angular momentum induced during an encounter divided by the time between encounters. Show that the torque exerted on the disc by the protoplanet is given by

$$\dot{J} = \frac{8(Gm_p)^2\Sigma R}{27\Omega^2 a_0^3}$$

where Σ is the disc surface density, and a_0 is the minimum impact parameter.

Question continues on next page.

For a protoplanet we take $a_0 = R(m_p/3M_*)^{1/3}$, where M_* is the mass of the central star. Show that gap formation will occur in the disc if

$$\frac{m_p}{M_*} > \frac{27\pi\nu}{8R^2\Omega}.$$

In deriving this result you may assume that the viscous torque acting on the inner disc edge is given by

$$\dot{J} = -3\pi\nu\Sigma R^2\Omega$$

where ν is the kinematic viscosity.

The kinematic viscosity is given by $\nu = \alpha H^2\Omega$ where H is the semi–thickness of the disc, and α is a constant. Conditions in protoplanetary discs are such that $H/R = 0.07$ and $\alpha = 6 \times 10^{-3}$, approximately. Estimate the required protoplanet/star mass ratio for gap formation to occur. Given that the masses of Jupiter, Saturn, and Uranus are 10^{-3}, 3×10^{-4}, and 4.8×10^{-5} $M_\odot$ respectively, comment on the implications of your answer for the formation of the giant planets in the solar system.

3) [*50 marks*] Write brief essays on the following topics, giving both a qualitative and quantitative account of the physical processes involved.

(*i*). *Planet formation in protostellar discs*:

Describe the characteristic of protostellar discs

Discuss dust grain evolution leading to the formation of larger solids, giving relevant time scales

Discuss the growth of planetesimals and the role of gravitational focussing in increasing growth rates

Discuss the gravitational interaction between protoplanets and protostellar discs, and their relevance to extrasolar planetary systems

(*ii*). *Accretion disc formation and evolution*:

Discuss the different means by which accretion discs can form

Describe the different classes of close binary systems, in particular those applying to semi-detached systems

Give estimates of the potential energy output that may be obtained by accretion onto different objects, and compare with other energy source in astrophysics

Discuss the requirement for an anomalous viscosity in accretion discs

Describe different angular momentun transfer mechanisms and the scenarios in which they are likely to play a role

(*iii*). *Accretion disc – magnetosphere interactions*:

Discuss the role of disc–magnetosphere interactions in affecting the inner regions of accretion discs

Describe the competing forces at work and the range of possible outcomes

Discuss how the rotation rates of the central stars may be modified by interaction between an accretion disc and stellar magnetosphere

Describe how a simple model of an accretion disc interacting with the magnetosphere of a neutron star may explain the origin of millisecond pulsars