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M.Sc. Astrophysics

ASTM003 Angular Momentum and Accretion in Astrophysics

Saturday 22 June 2002 10.00 - 11.30

The duration of this examination is one and a half hours.

Answer all of section A and one question from section B.

The use of a calculator in this examination is permitted, provided that you do not make use of any programming, graph-plotting or algebraic facilities that your calculator may have.

The following notation is used throughout unless otherwise stated: The pressure, density, surface density, and temperature are denoted by P, ρ, Σ, and T respectively. The magnetic field strength, current density, and effective temperature are denoted by $\mathbf{B}$, $\mathbf{j}$, and T_{eff}. The mean molecular weight, gas constant, and permittivity of free space are μ, $\mathcal{R}$, and μ_0 respectively.

The gravitational constant $G = 6.67\times10^{-11}$ Kg^{-1} m^3 s^{-2}. Stefan's constant $\sigma = 5.7\times10^{-8}$ J m^{-2} s^{-1} K^{-4}. The speed of light $c = 3\times10^8$ m s^{-1}. The solar mass $M_\odot = 2\times10^{30}$ Kg. The solar radius $R_\odot = 7\times10^8$ m.

The following mathematical identities may be useful.

The cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{i}}(A_jB_k - A_kB_j) - \hat{\mathbf{j}}(A_iB_k - A_kB_i) + \hat{\mathbf{k}}(A_iB_j - A_jB_i)$$

$\nabla\times\mathbf{A}$ in cylindrical polar coordinates:

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) - \hat{\phi}\left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right)$$

The divergence theorem:

$$\int_V \nabla.\mathbf{A}dV = \int_S \mathbf{A}.d\mathbf{S}$$

SECTION A - 50 Marks

1) The standard diffusion equation that describes the surface density evolution of a viscous accretion disc may be written:

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial j}{\partial R}\right)^{-1}\frac{\partial}{\partial R}\left(R^3 \nu \Sigma \frac{d\Omega}{dR}\right)\right] \tag{1}$$

where j is the specific angular momentum, Ω is the angular velocity, Σ is the surface density, and ν is the kinematic viscosity.

Show that in a steady-state Keplerian accretion disc the following expression holds:

$$\nu \Sigma = \frac{\dot{m}_d}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \tag{2}$$

where $\dot{m}_d = -2\pi R \Sigma v_R$ is the constant rate of inward mass flow, v_R is the radial velocity, and R_* denotes the position in the disc near to the central star where $d\Omega/dR = 0$. In deriving equation (2) you may use the expression

$$v_R \frac{\partial j}{\partial R} = \frac{1}{R\Sigma}\frac{\partial}{\partial R}\left[R^3 \nu \Sigma \frac{d\Omega}{dR}\right].$$

The region close to the star where the disc joins the stellar surface is known as the boundary layer. Explain briefly why this region is thought to provide a source of high energy photons that cannot be readily explained as arising from the main body of the disc. Provide one example of a scenario in which a boundary layer is thought to explain the emission of high energy photons.

2) The initial phase of planet formation is thought to involve the coagulation of dust grains that are initially well mixed with the gas in a protoplanetary disc. The grains subsequently sink towards the disc midplane to form a dense dust layer. The drag force experienced by a grain as it sinks is given by

$$F_{drag} = \pi a^2 \rho c_s v \tag{3}$$

where a is the grain radius, ρ is the gas density, c_s is the sound speed of the gas, and v is the vertical velocity of the grain. Show that for a thin disc in which $H \ll R$, a grain located at one scale height H above the disc midplane sinks at a velocity v_{term} given by

$$v_{term} = \frac{4a\rho_s \Omega^2 H}{3\rho c_s} \tag{4}$$

where Ω is the orbital angular velocity and ρ_s is the density of a single grain (density of solid material).

As a particle sinks through the disc it will collide with other grains that remain suspended in the gas. Assuming that such collisions always result in the grains sticking together, show that the rate at which the grain mass increases is given by the expression

$$\frac{dm_p}{dt} = \pi a_p^2 n_{gr} m_{gr} v_{term} \tag{5}$$

where n_{gr} is the number density of grains that remain suspended in the gas, m_{gr} is the mass of these grains, m_p is the mass of the accreting particle, and a_p is the radius of the accreting particle.

Further show that the particle mass when it reaches the midplane is given approximately by the expression

$$m_p = \frac{\pi}{3} \frac{(n_{gr} m_{gr} v_{term} t_{sink})^3}{16\rho_s^2} \tag{6}$$

where t_{sink} is the time required for the particle to reach the midplane. In deriving equation (6) you should assume that the density of the accreting particle remains constant as its mass increases, that the vertical velocity of the particle, v_{term}, remains constant, and that the initial particle mass is very much smaller than its final mass.

Question continues on next page.

Assuming that the mass ratio of gas to dust in a protoplanetary disc is 100, show that equation (6) may be approximated by the expression

$$m_p = \frac{\pi \Sigma^3}{4.8 \times 10^7 \rho_s^2} \tag{7}$$

where Σ is the disc surface density.

If the disc surface density at 1 Astronomical Unit from the central star is $\Sigma = 1 \times 10^4$ Kg m^{-2}, and the density of solid material $\rho_s = 3 \times 10^3$ Kg m^{-3}, estimate the final mass and radius of the accreting particle as it reaches the disc midplane.

3) Consider a rotating star that has a strong dipolar magnetic field. The magnetic field corotates with the star and threads through the inner regions of a surrounding circumstellar disc. The interaction between the rotating magnetic field and the more slowly rotating parts of the disc cause these regions of the disc to be repelled from the star, and may lead to the truncation of the disc inner edge at a radius R_{in}.

The magnetic force per unit volume acting in the disc may be written as

$$\mathbf{f} = \mathbf{j} \times \mathbf{B}$$

where $\mathbf{B}$ is the magnetic field strength and $\mathbf{j}$ is the current density. Ampére's law gives

$$\mathbf{j} = \frac{\nabla \times \mathbf{B}}{\mu_0}.$$

Working in cylindrical polar coordinates, assuming the system is axisymmetric, and that $B_r \ll B_z$ and $B_r \ll B_\phi$, show that torque per unit volume acting on the disc may be written

$$\mathcal{T} = \frac{R B_z}{\mu_0} \frac{\partial B_\phi}{\partial z}$$

where B_r, B_ϕ, and B_z are the r, ϕ, and z components of the magnetic field, respectively. Show that the total torque may be written

$$\dot{J} = \int_{-H}^{H} \int_{R_{in}}^{\infty} \frac{2\pi R^2 B_z}{\mu_0} \frac{\partial B_\phi}{\partial z} dR dz$$

where H is the disc semi–thickness.

Question continues on next page.

Using simple arguments, show that an estimate of the torque may be obtained in the form

$$\dot{J} = \frac{4\pi R_{in}^3}{3\mu_0}|\mathbf{B}_{in}|^2$$

where $\mathbf{B}_{in}$ is the magnetic field strength at the position R_{in}.

Here you may assume that the stellar magnetic field is such that

$$|\mathbf{B}(R)| = |\mathbf{B}_0|\left(\frac{R_*}{R}\right)^3,$$

where R_* is the stellar radius and $\mathbf{B}_0$ is a constant. Show further that the truncation radius of the disc, R_{in}, is given by

$$\frac{R_{in}}{R_*} = \left(\frac{4\pi|\mathbf{B_0}|^2 R_*^{5/2}}{3\sqrt{GM_*}\dot{m}_d\mu_0}\right)^{2/7}$$

where M_* is the mass of the star. In deriving this result you may assume that the viscous torque exerted at the inner edge of the disc is

$$\dot{J}_\nu = -3\pi\nu\Sigma R_{in}^2\Omega(R_{in})$$

and $\dot{m}_d = 3\pi\nu\Sigma$. Here, ν is the kinematic viscosity, Σ is the disc surface density, and $\dot{m}_d$ is the steady state mass flow rate through the disc.

SECTION B - Each question 50 marks

1) The virial theorem for a non–magnetised gaseous medium subject to internal pressure forces and self–gravity may be written as

$$\frac{d^2 I}{dt^2} = 4\mathcal{K} + 2E_g + 6\int_V P dV \tag{8}$$

where the moment of inertia, I, and kinetic energy $\mathcal{K}$ are given by

$$I = \int_V \rho r^2 dr, \qquad \mathcal{K} = \frac{1}{2}\int_V \rho v^2 dV,$$

r is the radius from the centre of mass (and origin of coordinate system), $v = |\mathbf{v}|$ where $\mathbf{v}$ is the velocity, P is the thermal pressure, and ρ is the density. E_g represents the gravitational potential energy which is given by:

$$E_g = -\frac{1}{2}G\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV dV'. \tag{9}$$

Derive equation (8), assuming only that the pressure goes to zero at the surface of the cloud. You may use the fact that the force per unit volume is given by

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla P - \rho\nabla\Phi \tag{10}$$

where the gravitational potential is given by

$$\Phi = -G\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'. \tag{11}$$

Consider an isothermal molecular cloud of mass M, rotating uniformly with angular velocity Ω, and within which the greatest distance between two points in the cloud is D. By assuming that the constituent gas obeys the ideal gas law, show that a sufficient condition for gravitational collapse of this cloud may be written as

$$\frac{GM}{D} > 2D^2\Omega^2 + \frac{6\mathcal{R}T}{\mu}.$$

Consider a molecular cloud of radius 3×10^{15} m and mass 4×10^{30} Kg uniformly rotating with angular velocity 10^{-14} radians s^{-1}. Estimate the size of the protostellar disc that would result from the collapse of this cloud.

2) The radiative flux, F_ν, emitted at a particular frequency, ν, by a black–body with effective temperature T_{eff} is given by

$$F_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/KT_{eff}\right) - 1}, \tag{12}$$

where h is Planck's constant, K is Boltzmann's constant, c is the speed of light. Consider an axisymmetric accretion disc which has a radial effective temperature profile given by

$$T_{eff} = T_I \left(\frac{R_I}{R}\right)^{-4/5}$$

between the disc inner radius R_{in} and an intermediate radius R_I, and has a radial effective temperature profile given by

$$T_{eff} = T_I \left(\frac{R_I}{R}\right)^{-6/7}$$

between the intermediate radius R_I and the outer disc radius R_{out}, where T_I is the constant temperature at radius R_I. The effective temperature at the disc inner edge is given by $T_{eff} = T_{in}$ and at the outer edge is given by $T_{eff} = T_{out}$. Show that the luminosity, L_ν, emitted at a particular frequency, ν, is given by the expression

$$\begin{aligned}\frac{1}{2}L_\nu = &\left(\frac{2\pi}{c}\right)^2 \frac{5}{4}K^{5/2}\nu^{1/2}T_I^{5/2}R_I^2 h^{-3/2}\int_0^{x_I}\frac{x^{3/2}}{\exp\left(x\right)-1}dx \\ &+\left(\frac{2\pi}{c}\right)^2 \frac{7}{6}K^{7/3}\nu^{2/3}T_I^{7/3}R_I^2 h^{-4/3}\int_{x_I}^{\infty}\frac{x^{4/3}}{\exp\left(x\right)-1}dx\end{aligned} \tag{13}$$

where $x = h\nu/(KT_{eff})$ and $x_I = h\nu/(KT_I)$. In deriving this expression, you should only consider emitted frequencies ν such that $h\nu \ll KT_{in}$ and $h\nu \gg KT_{out}$.

Make a sketch of $\log(L_\nu)$ versus $\log(\nu)$ for such a disc, and explain the shape of the curve. Your sketch should contain the regimes such that $h\nu > KT_{in}$ and $h\nu < KT_{out}$.

3) Write an essay that describes the formation and evolution of accretion discs, and their importance in astrophysics. You should discuss the full range of physical processes that occur in accretion discs in a quantitative and qualitative manner, using numerical estimates of physical quantities where relevant. Your essay should include discussion of the following important points:

- The range of scenarios in which accretion discs are thought to play an important role.
- The means by which accretion discs form.
- Constraints on the types of objects around which discs may be able to occur.
- Numerical estimates of the energy output from discs around different astronomical objects. These estimates should include a comparison with the energy output obtained from alternative sources such as nuclear fusion reactions, or the direct conversion of mass to energy.
- A discussion of the possible mechanisms of angular momentum transport in discs. You should describe under which conditions each mechanism may be appropriate, and explain why it has been suggested that an anomalously high viscosity operates in accretion discs.
- A discussion of the range of characteristic temperatures expected in accretion discs surrounding different astronomical objects.
- The spectral characteristics of accretion discs.
- The role of a boundary layer where the disc joins onto the central object, and its implications for observations of discs.
- Interaction between an accretion disc and magnetosphere of the central star.
- Planet formation in protostellar discs.