QUEEN MARY
UNIVERSITY OF LONDON
M.Sc. Astrophysics

ASTM003 Angular Momentum and Accretion in Astrophysics

Tuesday 22nd May 2001 14:30 – 16:00

The duration of this examination is one and a half hours.

Answer all of section A and one question from section B.

The use of a calculator in this examination is permitted, provided that you do not make use of any programming, graph-plotting or algebraic facilities that your calculator may have.

The following notation is used throughout unless otherwise stated: The pressure, density, surface density, and temperature are denoted by P, ρ, Σ, and T respectively. The effective temperature, mean molecular weight, gas constant, and kinematic viscosity are T_{eff}, μ, $\mathcal{R}$, and ν respectively.

The gravitational constant $G = 6.67 \times 10^{-11}$ Kg^{-1} m^3 s^{-2}. Stefan's constant $\sigma = 5.7 \times 10^{-8}$ J m^{-2} s^{-1} K^{-4}. The speed of light $c = 3 \times 10^8$ m s^{-1}. The solar mass $M_\odot = 2 \times 10^{30}$ Kg. The solar radius $R_\odot = 7 \times 10^8$ m. One Astronomical Unit (AU) $= 1.5 \times 10^{11}$ m. The gas constant $\mathcal{R} = 8.263 \times 10^3$ J K^{-1} Kg^{-1}.

[Turn over]

SECTION A - 50 Marks

A1) The virial theorem for an isolated fluid mass acting under the forces of self–gravity and pressure may be written

$$\frac{d^2I}{dt^2} = 4\mathcal{K} + 6\int_V PdV + 2E_g$$

where I is the moment of inertia, $\mathcal{K}$ is the total kinetic energy of the fluid mass, P is the pressure, and E_g is the total gravitational potential energy given by

$$E_g = -\frac{1}{2}G\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dVdV'$$

where $\rho(\mathbf{r})$ is the density at position $\mathbf{r}$ in the cloud.

Consider a spherical isothermal molecular cloud of mass M, rotating uniformly with angular velocity Ω, and with radius R. The cloud has internal turbulent motions such that the magnitude of the maximum turbulent velocity is $|v_{max}|$. By assuming that the constituent gas obeys the ideal gas law, show that a sufficient condition for gravitational collapse of this cloud may be written as

$$\frac{GM}{2R} > 2R^2\Omega^2 + 2|v_{max}|^2 + \frac{6\mathcal{R}T}{\mu}.$$

The molecular cloud collapses to form a protostar of mass M_s which contains most of the mass of the cloud, and is surrounded by a Keplerian protostellar disc. Derive an expression for the radius of the disc that forms in terms of the original radius of the cloud R, the original angular velocity of the cloud Ω, and the mass of the protostar M_s. Assuming values of $\Omega = 10^{-14}$ radians s^{-1}, $R = 3 \times 10^{15}$ m and $M_s = 6 \times 10^{30}$ Kg, calculate the size of the protostellar disc that forms.

A2) A Keplerian accretion disc in orbit around a central star of mass M_s is supported in the vertical direction by pressures forces that balance the vertical component of the gravitational force due to the star. Assuming that the gas in the disc is obeys the ideal gas equation of state and is isothermal, show that the resulting vertical density profile is given by

$$\rho(z) = \rho(0) \exp\left[-z^2/(2H^2)\right]$$

where z is the vertical coordinate, ρ is the density, and

$$H^2 = \frac{\mathcal{R}T}{\mu\Omega^2}$$

gives a measure of the disc vertical thickness.

The rate of viscous dissipation of energy per unit area in a Keplerian accretion disc is given by

$$\epsilon_D = \frac{9}{4}\Omega^2 \nu \Sigma$$

where Σ is the disc surface density and ν is the kinematic viscosity. Show that a steady state disc has a radial effective temperature profile given by

$$T_{eff} = \left(\frac{9}{8}\frac{GM_s\nu\Sigma}{R^3\sigma}\right)^{1/4}.$$

Assuming that the disc has a isothermal temperature profile in the vertical direction, with the temperature T being equal to T_{eff}, show that the ratio of disc thickness to radius H/R can be written as

$$\frac{H}{R} = \sqrt{\frac{\mathcal{R}}{\mu}}\left(\frac{9}{24\pi}\frac{\dot{m}}{(GM_s)^3\sigma}\right)^{1/8} R^{1/8}$$

where you may assume $\dot{m} = 3\pi\nu\Sigma$ is the mass accretion rate through the disc.

Protostellar discs are observed to have mass accretion rates of $\dot{m} = 10^{-8}$ $M_\odot$ per year ($\equiv 6.34 \times 10^{14}$ Kg s^{-1}). Estimate the thickness of a protostellar disc at a radius of 1 AU from a 1 $M_\odot$ T Tauri star, assuming that $\mu = 2$.

[Turn over]

A3) Consider a protoplanet of mass m_p forming within a gaseous, Keplerian protostellar disc at a radius R in orbit around a central star of mass M_s. Being embedded within the disc, the protoplanet is able to accrete gas from it. The protoplanet exerts a torque on the protostellar disc material that is given by

$$T_p = \frac{8(Gm_p)^2 \Sigma R}{27\Omega^2 \Delta^3}$$

where Δ denotes the radial distance between the protoplanet and the disc material on which it exerts the torque. This torque leads to the formation of an annular gap in the disc in the vicinity of the protoplanet's orbital radius. The viscous torque experienced by disc material located at the edge of the gap is given by

$$T_v = 2\pi\nu\Sigma R^3 \frac{d\Omega}{dR}.$$

Show that the width of the gap obtained is given by

$$\Delta = \left(\frac{8(Gm_p)^2}{81\pi\Omega^3 \nu R} \right)^{1/3}.$$

In order for the formation of a gap to affect the rate of gas accretion onto the protoplanet, the gap width must be larger than the protoplanet Roche lobe size given by

$$R_r = R\left(\frac{m_p}{3M_s}\right)^{1/3}.$$

Show that the condition

$$\frac{m_p}{M_s} > \frac{27\pi}{8}\alpha\left(\frac{H}{R}\right)^2$$

must be satisfied for gas accretion by a forming protoplanet to be substantially affected by gap formation. You may use the relation $\nu = \alpha H^2 \Omega$.

Protostellar discs are estimated to have $\alpha = 0.01$ and $H/R = 0.07$. Estimate the mass ratio required for gap formation to affect gas accretion onto a forming protoplanet in such a disc. Comment on the significance of your answer in the light of the fact that the Jupiter–Sun mass ratio is 10^{-3}, and the Earth–Sun mass ratio is 3×10^{-6}.

SECTION B - Each question 50 marks

B1) Consider an axisymmetric accretion disc with surface density Σ, kinematic viscosity ν, and in which forces due to pressure and self–gravity may be neglected. Derive the disc surface density evolution equation in the form

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial (R^2\Omega)}{\partial R}\right)^{-1}\frac{\partial}{\partial R}\left(R^3\nu\Sigma\frac{d\Omega}{dR}\right)\right] = 0$$

where R is the radial coordinate and Ω is the angular velocity. In doing so may assume that the torque acting in the direction of speeding the disc up, due to material interior to R, is given by

$$\mathcal{T} = -2\pi R^3 \nu\Sigma\frac{d\Omega}{dr}.$$

Show that if the disc is in a state of Keplerian rotation, the surface density evolution equation is

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(R^{1/2}\nu\Sigma\right)\right].$$

Deduce that in a steady state disc

$$\nu\Sigma = \frac{\dot{m}_d}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

where $\dot{m}_d = -2\pi R\Sigma v_R$ is the constant rate of inwards mass flow in the disc, v_R is the radial velocity, and R_* denotes the position in the disc where $d\Omega/dR = 0$.

The rate of energy production per unit area due to viscous dissipation in an accretion disc is

$$\epsilon_D = R^2\nu\Sigma\left(\frac{d\Omega}{dR}\right)^2.$$

Show that the effective temperature profile in a steady state accretion disc is given by

$$T_{eff}^4 = \frac{3GM\dot{m}_d}{8\pi R^3\sigma}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

where σ is Stefan's constant.

[Turn over]

B2) The radiative flux, F_ν, emitted at a particular frequency, ν, by a black–body with effective temperature T_{eff} is given by

$$F_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/KT_{eff}\right) - 1}, \tag{1}$$

where h is Planck's constant, K is Boltzmann's constant, c is the speed of light. Consider an axisymmetric accretion disc with a radial effective temperature profile given by

$$T_{eff} = \beta R^{-\alpha} \tag{2}$$

where α and β are positive constants. The effective temperature at the disc inner edge is given by $T_{eff} = T_{in}$ and at the outer edge is given by $T_{eff} = T_{out}$. Show that the luminosity, L_ν, emitted at a particular frequency, ν, is given by the expression

$$\frac{1}{2}L_\nu = \left(\frac{2\pi}{c}\right)^2 \frac{K^{2/\alpha}}{\alpha}\nu^{3-2/\alpha}\beta^{2/\alpha}h^{1-2/\alpha}\int_0^\infty \frac{x^{2/\alpha-1}}{\exp\left(x\right) - 1}dx, \tag{3}$$

where $x = h\nu/(KT_{eff})$. In deriving this expression, you should only consider emitted frequencies ν such that $h\nu \ll KT_{in}$ and $h\nu \gg KT_{out}$.

What does equation (3) tell you about the relation between the radial effective temperature distribution of an accretion disc and the emitted spectrum ?

A steady state accretion disc has $T_{eff} = \beta R^{-3/4}$. Make a sketch of $\log(L_\nu)$ versus $\log(\nu)$ for such a disc, and explain the shape of the curve. Your sketch should contain the regimes such that $h\nu > KT_{in}$ and $h\nu < KT_{out}$.

B3) Write an essay on the physics of accretion discs and their importance in astrophysics. Your discussion should include both a qualitative and a quantitative description of the relevant physical processes that may occur in accretion discs, and numerical estimates should be included where appropriate. Your essay should cover the following key points:

- The range of astrophysical phenomena in which accretion discs are thought to play an important role.
- The means by which accretion discs may form, and constraints of the types of objects around which discs may be able to occur.
- Numerical estimates of the energy output from discs around different astronomical objects. These estimates should include a comparison with the energy output obtained from alternative sources such as nuclear fusion reactions, or the direct conversion of mass to energy.
- A discussion of the possible mechanisms of angular momentum transport in discs. You should describe under which conditions each mechanism may be appropriate, and explain why it has been suggested that an anomalously high viscosity operates in accretion discs.
- A discussion of the range of characteristic temperatures expected in accretion discs surrounding different astronomical objects. The spectral characteristics of accretion discs should also be described.
- The role of a boundary layer where the disc joins onto the central object, and its implications for observations of discs, should be described.

End of Examination

R.P. Nelson