QUEEN MARY AND WESTFIELD COLLEGE
UNIVERSITY OF LONDON
M.Sc. Astrophysics

ASTM003 Angular Momentum and Accretion in Astrophysics

Saturday 15 July 2000 10.00 - 11.30

The duration of this examination is one and a half hours.

Answer all of section A and one question from section B.

The use of a calculator in this examination is permitted, provided that you do not make use of any programming, graph-plotting or algebraic facilities that your calculator may have.

The following notation is used throughout unless otherwise stated: The pressure, density, surface density, and temperature are denoted by P, ρ, Σ, and T respectively. The magnetic field strength, current density , and effective temperature are denoted by $\mathbf{B}$, $\mathbf{j}$, and T_{eff}. The mean molecular weight, gas constant, kinematic viscosity, and permittivity of free space are μ, $\mathcal{R}$, ν, and μ_0 respectively.

The gravitational constant $G = 6.67\times10^{-11}$ Kg^{-1} m^3 s^{-2}. Stefan's constant $\sigma = 5.7\times10^{-8}$ J m^{-2} s^{-1} K^{-4}. The speed of light $c = 3\times10^8$ m s^{-1}. The solar mass $M_\odot = 2\times10^{30}$ Kg. The solar radius $R_\odot = 7\times10^8$ m.

The following mathematical identities may be useful.

The cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{i}}(A_jB_k - A_kB_j) - \hat{\mathbf{j}}(A_iB_k - A_kB_i) + \hat{\mathbf{k}}(A_iB_j - A_jB_i)$$

$\nabla\times\mathbf{A}$ in cylindrical polar coordinates:

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) - \hat{\phi}\left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right)$$

Trigonometric Identities:

$$\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

SECTION A - 50 Marks

1) The virial theorem for an isolated fluid mass (e.g. a molecular cloud) acting under the forces of self–gravity and pressure may be written

$$\frac{d^2 I}{dt^2} = 4\mathcal{K} + 6\int_V P dV + 2E_g$$

where I is the moment of inertia, $\mathcal{K}$ is the total kinetic energy of the fluid mass, P is the pressure, and E_g is the total gravitational potential energy given by

$$E_g = -\frac{1}{2}G\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dVdV'$$

where $\rho(\mathbf{r})$ is the density at position $\mathbf{r}$ in the cloud.

Consider an isothermal molecular cloud of mass M, rotating uniformly with angular velocity Ω, and within which the greatest distance between two points in the cloud is D. By assuming that the constituent gas obeys the ideal gas law, show that a sufficient condition for gravitational collapse of this cloud may be written as

$$\frac{GM}{D} > 2D^2\Omega^2 + \frac{6\mathcal{R}T}{\mu}.$$

Consider a molecular cloud of radius 3×10^{15} m and mass 4×10^{30} Kg uniformly rotating with angular velocity 10^{-14} radians s^{-1}. Estimate the size of the protostellar disc that would result from the collapse of this cloud.

2) Consider the growth of a large, gravitating, rocky planetary core by the accretion of smaller background solid objects during planet formation. Let the mass of the core be m_c, the core radius be R_c, the background object mass be m, and the number density of background objects be n. The accretion rate onto the core is given by

$$\frac{dm_c}{dt} = nmv\pi R_c^2 \left(1 + \frac{2Gm_c}{R_c v^2}\right)$$

where v is the velocity dispersion of the background objects. Using the principles of angular momentum and energy conservation, derive the gravitational focusing factor contained in brackets on the right hand side of the above equation.

Consider the situation where $Gm_c/R_c \gg v^2$. Show that the accretion rate by the planetary core in this case can be approximated by

$$\frac{dm_c}{dt} = \frac{2nm\pi G}{v} \left(\frac{3}{4\pi\rho_c}\right)^{1/3} m_c^{4/3}$$

where ρ_c is the density of the accreting core.

Show that the time required for the planetary core to grow from some initial mass m_{c0} to a very much larger mass is given by

$$t_g = \frac{3m_{c0}^{-1/3} v}{2\pi Gnm} \left(\frac{4\pi\rho_c}{3}\right)^{1/3}$$

Explain briefly how estimates of the planetary growth time obtained from this last equation may be used to constrain theories of planet formation.

3) A rotating star has a strong, dipolar magnetic field that corotates with the star. The magnetic field threads through a surrounding circumstellar disc out to some radius. The interaction between the rotating magnetic field and the more slowly rotating parts of the disc cause these regions to be repelled from the star. This process may lead to the truncation of the disc inner edge at some distance, R_{in}, from the star.
The magnetic force per unit volume acting on the disc is given by

$$\mathbf{f} = \mathbf{j} \times \mathbf{B}$$

where $\mathbf{B}$ is the magnetic field strength and $\mathbf{j}$ is the current density. Ampére's law gives

$$\mathbf{j} = \frac{\nabla \times \mathbf{B}}{\mu_0}.$$

Working in cylindrical polar coordinates, assuming that the system is axisymmetric, and that $B_r \ll B_z$ and $B_r \ll B_\phi$, show that torque per unit volume acting on the disc may be written

$$\mathcal{T} = \frac{RB_z}{\mu_0}\frac{\partial B_\phi}{\partial z}$$

where B_r, B_ϕ, and B_z are the r, ϕ, and z components of the magnetic field, respectively.
Hence show that the total torque becomes

$$\dot{J} = \int_{-H}^{H}\int_{R_{in}}^{\infty} \frac{2\pi R^2 B_z}{\mu_0}\frac{\partial B_\phi}{\partial z} dRdz$$

where H is the disc semi–thickness.
Using simple arguments, show that an estimate of the torque is given by

$$\dot{J} = \frac{4\pi R_{in}^3}{3\mu_0}|\mathbf{B}_{in}|^2$$

where $\mathbf{B}_{in}$ is the magnetic field strength at the position R_{in}.

Question continues on next page.

Here you may assume that the stellar magnetic field is such that

$$|\mathbf{B}(R)| = |\mathbf{B}_0| \left(\frac{R_*}{R}\right)^3,$$

where R_* is the stellar radius and $\mathbf{B}_0$ is a constant. Show further that the truncation radius of the disc, R_{in}, is given by

$$\frac{R_{in}}{R_*} = \left(\frac{4\pi|\mathbf{B_0}|^2 R_*^{5/2}}{3\sqrt{GM_*}\dot{m}_d\mu_0}\right)^{2/7}$$

where M_* is the mass of the star. In deriving this result you may assume that the viscous torque exerted at the inner edge of the disc is

$$\dot{J}_\nu = -3\pi\nu\Sigma R_{in}^2 \Omega(R_{in})$$

and $\dot{m}_d = 3\pi\nu\Sigma$. Here, ν is the kinematic viscosity, Σ is the disc surface density, and $\dot{m}_d$ is the steady state mass flow rate through the disc.

SECTION B - Each question 50 marks

1) Consider an axisymmetric accretion disc with surface density Σ, kinematic viscosity ν, and in which forces due to pressure and self–gravity may be neglected. Derive the disc surface density evolution equation in the form

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left[\left(\frac{\partial (R^2\Omega)}{\partial R}\right)^{-1}\frac{\partial}{\partial R}\left(R^3\nu\Sigma\frac{d\Omega}{dR}\right)\right] = 0$$

where R is the radial coordinate and Ω is the angular velocity.

You may assume that the torque acting in the direction of speeding the disc up, due to material interior to R, is given by

$$\mathcal{T} = -2\pi R^3\nu\Sigma\frac{d\Omega}{dr}.$$

Show that if the disc is in a state of Keplerian rotation, the surface density evolution equation is

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(R^{1/2}\nu\Sigma\right)\right].$$

Deduce that in a steady state disc

$$\nu\Sigma = \frac{\dot{m}_d}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

where $\dot{m}_d = -2\pi R\Sigma v_R$ is the constant rate of inwards mass flow in the disc, v_R is the radial velocity, and R_* denotes the position in the disc where $d\Omega/dR = 0$.

The rate of energy production per unit area due to viscous dissipation in an accretion disc is

$$\epsilon_D = R^2\nu\Sigma\left(\frac{d\Omega}{dR}\right)^2.$$

Show that the effective temperature profile in a steady state accretion disc is given by

$$T_{eff}^4 = \frac{3GM\dot{m}_d}{8\pi R^3\sigma}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

where σ is Stefan's constant.

2) Consider a protoplanet of mass m_p on a circular orbit in an axisymmetric Keplerian disc at radius R. The disc material interacts with the protoplanet as it shears past it, leading to angular momentum exchange. Consider only those parts of the disc that lie exterior to the protoplanet position. The interaction between disc and protoplanet can be described using the local impulse approximation. Here a Cartesian coordinate system is used that is centred on, and corotates with, the protoplanet.

The unperturbed disc flow past the the protoplanet occurs in the y direction only, with

$$v_y = u = -R\frac{d\Omega}{dR}x,$$

where Ω is the disc angular velocity at the protoplanet position. The equation of motion governing the deflection experienced by disc material due to an encounter with the protoplanet is

$$\frac{dv_x}{dt} = -\frac{Gm_p a}{(a^2+y^2)^{3/2}}$$

where v_x is the x component of the velocity and a is the impact parameter of the unperturbed disc material with respect to the protoplanet. Show that the value of v_x after the encounter is given by

$$v_x^2 = 4\left(\frac{Gm_p}{ua}\right)^2.$$

The angular momentum exchanged per unit mass during an encounter with the protoplanet is $R\Delta v_y$, where Δv_y is the change in the y component of the velocity. The rate of change of angular momentum of disc material is the change in angular momentum induced during an encounter divided by the time between encounters. Show that the torque exerted on the disc by the protoplanet is given by

$$\dot{J} = \frac{8(Gm_p)^2\Sigma R}{27\Omega^2 a_0^3}$$

where Σ is the disc surface density, and a_0 is the minimum impact parameter.

Question continues on next page.

For a massive protoplanet, $a_0 = R(m_p/3M_*)^{1/3}$, where M_* is the mass of the central star. Show that gap formation will occur in the disc if

$$\frac{m_p}{M_*} > \frac{27\pi\nu}{8R^2\Omega}.$$

In deriving this result you may assume that the viscous torque acting on the inner disc edge is given by

$$\dot{J} = -3\pi\nu\Sigma R^2\Omega$$

where ν is the kinematic viscosity.

The kinematic viscosity is given by $\nu = \alpha H^2\Omega$ where H is the semi–thickness of the disc, and α is a constant. Conditions in protoplanetary discs are such that $H/R = 0.07$ and $\alpha = 6 \times 10^{-3}$, approximately. Estimate the required protoplanet/star mass ratio for gap formation to occur. Given that the masses of Jupiter, Saturn, and Uranus are 10^{-3}, 3×10^{-4}, and 4.8×10^{-5} $M_\odot$ respectively, comment on the implications of your answer for the formation of the giant planets in the solar system.

3) Write an essay on the physics of accretion discs and their importance in astrophysics. Your discussion should include both a qualitative and a quantitative description of the relevant physical processes that may occur in accretion discs, and numerical estimates should be included where appropriate. Your essay should cover the following key points:

- The range of astrophysical phenomena in which accretion discs are thought to play an important role.
- The means by which accretion discs may form, and constraints of the types of objects around which discs may be able to occur.
- Numerical estimates of the energy output from discs around different astronomical objects. These estimates should include a comparison with the energy output obtained from alternative sources such as nuclear fusion reactions, or the direct conversion of mass to energy.
- A discussion of the possible mechanisms of angular momentum transport in discs. You should describe under which conditions each mechanism may be appropriate, and explain why it has been suggested that an anomalously high viscosity operates in accretion discs.
- A discussion of the range of characteristic temperatures expected in accretion discs surrounding different astronomical objects. The spectral characteristics of accretion discs should also be described.
- The role of a boundary layer where the disc joins onto the central object, and its implications for observations of discs, should be described.