Appendix D: Example Problems

Problem 1: Question

Suppose some category of galaxies has an observed surface brightness profile $I(R) = I_0\, f(R/R_0)$ with all galaxies having the same I_0 and function f but different galaxies having different R_0. If the mass-to-light ratio is constant everywhere then show that

$$L \propto v^4$$

where L is the total luminosity and v is a characteristic velocity.

Problem 2: Question

An astronomer performs optical spectroscopy on a faint galaxy. The spectrum shows a strong continuum with absorption lines and some emission lines superimposed. What is the morphological type of the galaxy?

The astronomer observes a second galaxy and finds very strong emission lines superimposed on a continuum and some absorption lines. How does the star formation rate of this second galaxy compare with that in the first? What morphological type might the second galaxy be?

Problem 3: Question

What is the difference between collisional and collisionless interactions between two massive bodies? Are the interactions between two gas clouds collisional or collisionless?

Two small systems of stars collide. In this particular case, the gravitational field of any one star changes the motions of other nearby stars substantially. Is the interaction between the two systems collisional or collisionless? How would this compare with the stars in two galaxies that merge together?

A large stationary gas cloud collapses under its own gravitation. 20 % of the initial potential energy is converted into heat, raising the temperature of the gas. Is this collapse dissipative or dissipationless?

Appendix D: Example Problems

Problem 4: Question

The Plummer potential has a gravitational potential $\Phi(r)$ at a distance r from the centre of a spherically-symmetric mass distribution that is given by

$$\Phi(r) = -\frac{GM_{tot}}{\sqrt{r^2 + a^2}} ,$$

where M_{tot} is the total mass, G is the constant of gravitation, and a is a constant. Derive from this an expression for the mass $M(r)$ interior to a radius r, and show that the density $\rho(r)$ at a radius r is

$$\rho(r) = \frac{3M_{tot}}{4\pi} \frac{a^2}{(r^2 + a^2)^{\frac{5}{2}}} .$$

Problem 5: Question

A family of radial density profiles $\rho(r)$ that have been popular for the theoretical modelling of spherically symmetric galaxies have been Dehnen models. These are defined so that the density profiles are

$$\rho(r) = \frac{q\, a}{4\pi} \frac{r^q}{r^3\,(r + a)^{q+1}} M_{tot} ,$$

where r is the radial distance from the centre of the galaxy, q is an adjustable parameter, a is a scaling constant (determining the size of the galaxy), and M_{tot} is the total mass. The special case of $q = 1$, which is called the Jaffe model, is particularly important because it is found to fit the observed $I(R)$ of ellipticals at least as well as the de Vaucouleurs $R^{1/4}$ profile.
What is the mass $M(r)$ interior to a radius r for any value of q?
What is the gravitational potential of a mass distribution having a Jaffe $\rho(r)$?
The Dehnen models have an interesting limit as $q \to 0$. What is it?
You may use the standard integral

$$\int \frac{r^{q-1}}{(r + a)^{q+1}}\, \mathrm{d}r = \frac{1}{q\, a} \frac{r^q}{(r + a)^q} + \text{constant} .$$

Hints

These questions involve calculations relating to spherically symmetric potentials. Under spherical symmetry, the gravitational potential Φ, the mass interior to a radius and the density ρ are all functions of the radius r from the centre alone. Converting between $\Phi(r)$ and the density $\rho(r)$ can be done using the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho$$

Appendix D: Example Problems

Problem 6: Question

A self-gravitating, spherically-symmetric cloud of gas has the same temperature T throughout. The equation of hydrostatic equilibrium gives, at a radial distance r from the centre of the cloud,

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} \;=\; -\,\frac{GM(r)}{r^2}\;,$$

where $P(r)$ is the gas pressure, $\rho(r)$ is the density of the gas, $M(r)$ is the mass interior to the radius r, and G is the constant of gravitation. The gas obeys the ideal gas equation, $P = n_p k_B T = \rho k_B T/m_p$ at any point, where $n_p(r)$ is the number density of gas particles, m_p is the mean mass of the gas particles, and k_B is the Boltzmann constant. The gas has the same chemical composition throughout.
Obtain a second-order differential equation involving ρ and r as the only variables.
Show that $\rho(r) = \sigma^2/2\pi G r^2$ is a solution to this differential equation where $\sigma = \sqrt{k_B T/m_p}$.

Problem 7: Question

The gravitational potential in a spherically-symmetric galaxy is given by

$$\Phi(r) \;=\; 2\pi G\rho_0 a^2\left(\ln(r^2+a^2) \;+\; \frac{2a}{r}\tan^{-1}\left(\frac{r}{a}\right)\right) \;+\; \Phi_0$$

at a distance r from the centre, where G is the constant of gravitation, and ρ_0, a and Φ_0 are constants.
What is the mass $M(r)$ interior to the radius r as a function of radius r?
What is the circular velocity v_{circ} as a function of radius r? What happens to v_{circ} when $r \gg a$? How does this compare with real galaxies?
What is the density ρ as a function of radius r? Do you recognise this density profile? How would you interpret the constants ρ_0 and a?
What practical constraint is there on the constant Φ_0? How does the potential $\Phi(r)$ behave as $r \to \infty$ and is this physically realistic?

Appendix D: Example Problems

Problem 8: Question

The Navarro-Frenk-White density profile is often used to represent galaxies. In this profile the density ρ at a distance r from the centre of the galaxy is given by

$$\rho(r) \quad = \quad \frac{k}{r(r+a)^2} \quad ,$$

where k and a are constants. What is the mass $M(r)$ interior to a radius r implied by this profile?
What happens to $M(r)$ as $r \to \infty$?

What is the gravitational potential Φ at a radius r?
What is the central density implied by this profile? Is it physically realistic?

Problem 9: Question

The region around the nucleus of a galaxy is observed to consist of a dense cluster of 10^7 stars moving in randomly orientated orbits with typical velocities of $100\,\mathrm{km\,s^{-1}}$. The radius of this region is observed to be $70\,\mathrm{pc}$.

Assuming the density of stars is uniform across the cluster, estimate the relaxation time of the stellar motions.

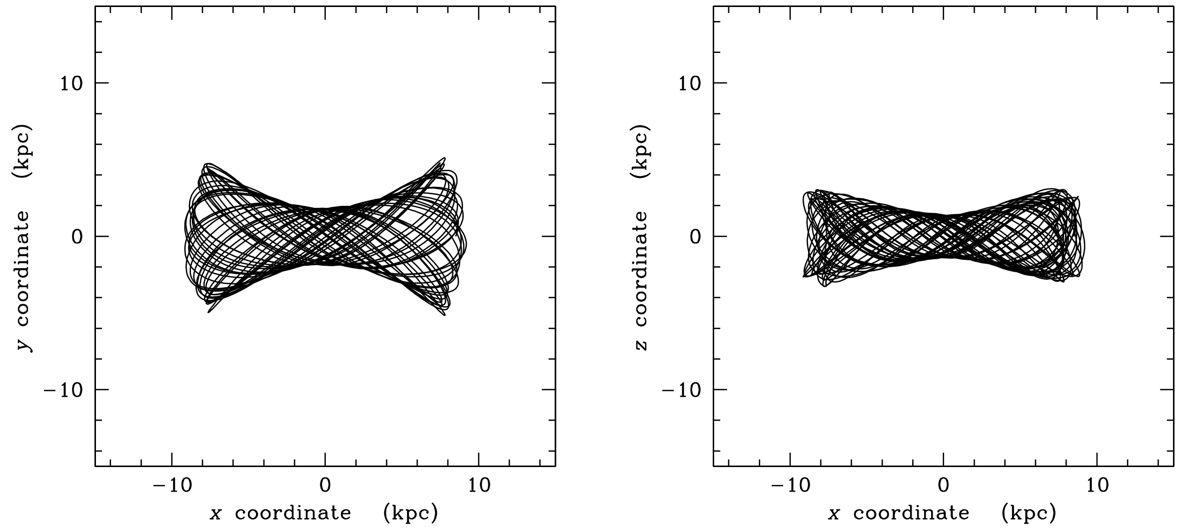
How does the relaxation time compare with the age of the galaxy? Can the dynamics of the stars around the nucleus be modelled as a collisionless system over the lifetime of the galaxy? How does this compare with the stars in regions away from the nucleus?

[The constant of gravitation is $G = 6.673 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}s^{-2}}$. The mass of the Sun is $1\ M_\odot = 1.989 \times 10^{30}$ kg. One year is 3.1557×10^7 s. The age of the Universe is about 13.7 Gyr.]

Appendix D: Example Problems

Problem 10: Question

The figures below shows the orbit of a star in a galaxy from two different perspectives. Is the galaxy's potential (i) spherically symmetric, (ii) oblate, or (iii) triaxial? Justify your answer on the basis of the character of the orbit.



Summarise briefly the properties of the orbits of stars in such a potential.

Problem 11: Question

The distribution function f in a spherically-symmetric galaxy is related to the mass density $\rho(r)$ at a radial distance r from the centre by

$$\rho(r) \quad = \quad 4\pi\,\sqrt{2}\,\overline{m}\,\int_{\Phi(r)}^{0} \sqrt{E_m - \Phi(r)}\; f(E_m)\,\mathrm{d}E_m \quad ,$$

where E_m is the energy per unit mass for a star, $\Phi(r)$ is the gravitational potential at a radius r, and $\overline{m}$ is the mean mass per star. Show that a functional form $f(E_m) = b\,(-E_m)^{\frac{7}{2}}$ is a solution to this equation for a Plummer potential, where b is a constant, using the potential and density given in Question 4. Express b is terms of G, M_{tot} and a using the result of Question 4. The substitution $E_m = \Phi\cos^2\theta$ and the standard result

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\;\cos^8\theta\;\mathrm{d}\theta \quad = \quad \frac{7\pi}{512}$$

may prove useful.

Assuming $\overline{m} = 0.70 M_\odot$, what is the value of the distribution function f for $(x, y, z, v_x, v_y, v_z) = (10\,\mathrm{kpc}, 0, 0, 0, 0, 200\,\mathrm{kms}^{-1})$ in a galaxy having a Plummer potential with a softening parameter $a = 1.70$ kpc and a total mass of $2.0 \times 10^{12} M_\odot$? Note that $x = y = z = 0$ corresponds to the centre of the galaxy in this coordinate system.

[$1 M_\odot = 1.989 \times 10^{30}$ kg, $1\,\mathrm{kpc} = 3.0857 \times 10^{19}$ m, and $G = 6.673 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$.]

Appendix D: Example Problems

Problem 12: Question

A spherical elliptical galaxy has a total density distribution

$$\rho_{\rm tot}(r) = \frac{\rho_0}{1 + r^2/a^2} \quad ,$$

as a function of radial distance r from its centre, where ρ_0 and a are constants (here the total density means the density including all stars, gas and dark matter). Show that the mass $M(r)$ interior to a radius r has the form $M(r) \propto r^3$ for $r \ll a$ and $M(r) \propto r$ for $r \gg a$.

Consider a population of massless test particles in the potential of this galaxy. Assume that this population is spherical, non-rotating, isothermal and isotropic, with velocity dispersion σ in each velocity component. What is the radial density distribution $\rho_p(r)$ of this test particle population, expressed in terms of $M(r)$ and r?

Solve for $\rho_p(r)$ in terms of r explicitly for large radii (i.e. for regions where $r \gg a$) to show that the density has a power law dependence on radius. What is the index of this power law? Give a physical interpretation of this index. What is the condition for the density distributions of the test particle population and the galaxy itself to have similar forms at large r?

Problem 13: Question

Many of the researchers who perform N-body simulations do so to study the dynamics of galaxies, but some others us N-body techniques to study the dynamics of globular clusters. Naively, we might expect the latter group of people would have an easier job, because they can easily afford as many particles in their simulations as there are stars in the real objects, and they do not need to worry about gas dynamics. We might therefore expect that globular cluster dynamics would be a well-understood subject by now. However, many problems have not been solved fully and plenty of difficult research remains to be done. This problem is to work out why.

Consider a globular cluster and a galaxy, both $\sim 10^{10}$ yr old. The globular cluster has a size ~ 20 pc across and contains 10^5 stars moving with a typical velocity $15\,{\rm km\,s^{-1}}$. The galaxy is ~ 20 kpc across and contains $\sim 10^{11}$ stars with a typical velocity $200\,{\rm km\,s^{-1}}$. Both of these are simulated using 10^5 particles. Give two reasons why the globular cluster simulation will be more difficult.

Appendix D: Example Problems

Problem 14: Question

The second Jeans equation in a spherically-symmetric potential gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(n\,\langle v_r^2\rangle \Big) \;+\; \frac{n}{r}\,\Big[\, 2\langle v_r^2\rangle \,-\, \langle v_\theta^2\rangle \,-\, \langle v_\phi^2\rangle \Big] \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\ ,$$

for a spherical polar coordinate system (r,θ,ϕ), where $n(r)$ is the number density of stars in space, $\Phi(r)$ is the gravitational potential, and v_r, v_θ and v_z are the components of the velocity in the r,θ and ϕ directions at a point.
A galaxy has a gravitational potential at a distance r from its centre that is given by

$$\Phi(r) \;=\; -\,\frac{GM_{tot}}{\sqrt{r^2+a^2}}\ ,$$

where M_{tot} is the total mass, a is a constant and G is the constant of gravitation. Assuming that the velocity dispersion σ of a population of stars is isotropic and constant over the whole galaxy, and that there is no net rotation, show that the number density of these stars in this potential is

$$n(r) \;=\; n_0\ \exp\left[\frac{GM_{tot}}{a\,\sigma^2}\left(\frac{1}{\sqrt{1+r^2/a^2}} \,-\, 1\right)\right]\ ,$$

where $n_0 = n(0)$.

Problem 15: Question

Observations of an isolated H II region show that the total flux of photons from the nebula through the Hα emission line at the top of the Earth's atmosphere is 9×10^6 photons $\mathrm{s^{-1}\,m^{-2}}$, while it is 6×10^6 photons $\mathrm{s^{-1}\,m^{-2}}$ in the Hβ line, 4×10^6 photons $\mathrm{s^{-1}\,m^{-2}}$ in Hγ, 3×10^6 photons $\mathrm{s^{-1}\,m^{-2}}$ in Hδ, and 5×10^6 photons $\mathrm{s^{-1}\,m^{-2}}$ in all other Balmer emission (line and continuum). The H II region lies at a distance of 900 pc from the Earth. What is the total luminosity of ultraviolet photons with wavelengths shorter than 912 Å from stars inside the H II region?

[1 pc = 3.0857×10^{16} m.]

Problem 16: Question

Observations of a part of the interstellar medium of the Galaxy show that a region of hot ionised gas (with a temperature 500 000 K, number density of ions 6000 $\mathrm{m^{-3}}$), a region of cold neutral gas (temperature 50 K, number density of molecules 2×10^7 $\mathrm{m^{-3}}$), and a region of warm neutral gas (temperature 10 000 K, number density of atoms 1×10^5 $\mathrm{m^{-3}}$) are in contact with each other. Which, if any, of these are in pressure equilibrium with the others?

Appendix D: Example Problems

Problem 17: Question

A star lying in the Galactic plane is observed to have a visual magnitude of $V = 13.60$ mag and a colour index $(B - V) = 0.98$ mag. Its spectrum shows it to be a dwarf star of spectral type G6 with a solar composition. Stars of this type are known to have an intrinsic colour of $(B - V)_0 = 0.76$ mag and an absolute visual magnitude of $M_V = +5.20$.
What is the extinction by the interstellar medium in the V band between us and the star? What is the distance of the star? What is the mean extinction per unit distance in the direction of the star expressed in mag kpc^{-1} for the V band? Will this extinction per unit distance be the same for other stars in the sky?
What would you expect the extinction to be towards the star in the I and K photometric bands (which have central wavelengths of 790 nm and 2.2 μm respectively)?

Problem 18: Question

The mean density in the form of stars in the disc of the Galaxy is observed to vary with the distance z from the Galactic plane as $\rho_s(z) = \rho_{so} \mathrm{e}^{-|z|/h_s}$ close to the Sun, where ρ_{so} is the density of stars in space in the plane, and h_s is a scale height (ρ_{so} and h_s are therefore constants at the distance of the Sun from the Galactic Centre). The density of the interstellar gas ρ_g is also found to vary exponentially with height, with $\rho_g(z) = \rho_{go} \mathrm{e}^{-|z|/h_g}$, where ρ_{go} and h_g are constants. Observations show that $h_s = 250$ pc and $h_g = 150$ pc and $\rho_{so} = 6\, \rho_{go}$. What is the ratio of the surface density of stars, Σ_s, to that of gas, Σ_g, at the Sun's distance from the Galactic Centre?
How do you expect the surface density of the dust, Σ_d, to compare with Σ_s?

Problem 19: Question

Observations of the extinction of starlight by dust show that the ratio of colour excesses, $E_{\mathrm{U-B}}/E_{\mathrm{B-V}}$, for the (U–B) and (B–V) colour indices is nearly constant across the Galaxy, regardless of the strength of the extinction. Prove that the parameter

$$Q \equiv (\mathrm{U-B}) - \frac{E_{\mathrm{U-B}}}{E_{\mathrm{B-V}}} (\mathrm{B-V})$$

is independent of interstellar extinction.
(Note that U, B and V are magnitudes in the near ultraviolet, in the blue and in the visual parts of the spectrum.)

A hot main sequence star in the Galactic plane is observed to have magnitudes U = 12.35, B = 12.69 and V = 12.00. What is the Q parameter for this star given that a standard value for $E_{\mathrm{U-B}}/E_{\mathrm{B-V}}$ is 0.72 in the Galaxy?

What is the spectral type of the star given the standard relationship between Q and spectral type below. What is the (B-V) colour excess $E_{\mathrm{B-V}}$ of the star? Estimate the

V-band interstellar extinction A_V and the star's distance.

The table below gives the spectral type, the intrinsic (B–V) colour index, the Q parameter and the absolute magnitude in the V band for hot main sequence stars.

Spectral type	O5V	B0V	B5V	A0V
$(B–V)_0$	−0.35	−0.31	−0.16	0.00
Q	−0.90	−0.84	−0.43	0.00
M_V	−5.8	−4.1	−1.1	+0.7

Appendix D: Example Problems

Problem 20: Question

How does the metallicity Z behave formally as the gas fraction $\mu \to 0$ in the Simple Model of chemical evolution? Is this realistic?

Show that in the Simple Model, the mean metallicity of a population of long-lived stars is given by

$$\langle Z \rangle \;=\; p\left(\frac{M_{\rm gas}(0)}{M_{\rm stars}} - 1\right)\ln\left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)}\right) \;+\; p\ ,$$

where $M_{\rm gas}(t)$ is the mass of gas in the volume at time t, $M_{\rm stars}(t)$ is the mass of stars at time t, and p is the yield.
Show that this can be written as $\langle Z \rangle \;=\; p\left(1 + \dfrac{\mu\ln\mu}{1-\mu}\right)$ in terms of the gas fraction μ.
What is the mean metallicity $\langle Z \rangle$ when the gas fraction $\mu \to 0$ as gas is used up entirely in star formation?

You may find helpful the standard integral

$$\int \ln(1 - x/a)\,{\rm d}x \;=\; (x-a)\ln(1-x/a) - x + {\rm constant}.$$

[Hint: the mean metallicity can be represented as $\langle Z \rangle \;=\; \dfrac{\int_0^{M_{\rm stars}} Z\,{\rm d}M'_{\rm stars}}{\int_0^{M_{\rm stars}} {\rm d}M'_{\rm stars}}$.]

Problem 21: Question

One variant on the Simple Model of galactic chemical evolution is the 'leaky-box' model. This simulates the effect of shocks from supernovae and winds from young massive stars by allowing gas to leave the box at a rate proportional to the star formation rate. Therefore the change $\delta M_{\rm total}$ in the total mass $M_{\rm total}$ in the box is

$$\delta M_{\rm total} \;=\; -\,c\,\delta M_{\rm stars}\ ,$$

where $\delta M_{\rm stars}$ is the change in the mass in stars, and c is a constant or proportionality. Use this to derive an expression for the mass in gas $M_{\rm gas}(t)$ at time t in terms of $M_{\rm total}(0)$ and $M_{\rm stars}(t)$.
Now modify the closed-box relation between $\delta M_{\rm metals}$ and $\delta M_{\rm stars}$ by adding an appropriate leaking term.
Use these two expressions to derive

$$\delta Z = \frac{p\,\delta M_{\rm stars}}{M_{\rm total}(0) - (1+c)\,M_{\rm stars}}\ .$$

This expression shows that the leaky box model won't solve the G-dwarf problem? Why?

Appendix D: Example Problems

Problem 22: Question

For gravitational lensing, for very distant sources (i.e., $D_S \gg D_L$), we can write the expression for the Einstein angular radius as

$$\theta_E \;=\; k\,\sqrt{M/D_L}\;,$$

where k is a constant. Find the value of k in arcsec if M is measured in solar masses and D_L in parsecs.

Problem 23: Question

An optical microlensing survey images a star field in the Galactic bulge close to the Galactic centre. Assuming that the dark matter halo is made from compact objects with approximately stellar masses and has a density distribution

$$\rho(r) \;=\; \frac{\rho_0\, b^2}{r^2 + b^2}\;,$$

where r is the radial distance from the Galactic centre, ρ_0 is the central dark matter density and b is a constant, derive an expression for the optical depth of microlensing to the field in terms of ρ_0, b and R_0. Express the result in terms of the distance R_0 of the Sun from the Galactic centre. You may assume that the star field is not significantly affected by dust extinction for this calculation.
Calculate τ if $R_0 = 8.0\,\mathrm{kpc}$, $b = 2.0\,\mathrm{kpc}$ and $\rho_0 = 2.0 \times 10^{-20}\,\mathrm{kg\,m^{-3}}$.

The standard result $\int r^2/(r^2+b^2)\,\mathrm{d}r \;=\; r \,-\, b\tan^{-1}(r/b) + \mathrm{constant}$, may prove useful.

Problem 24: Question

A weakly-interacting massive particle (WIMP) with a mass of $1000\,m_p$, where $m_p = 1.6726 \times 10^{-27}$ kg is the mass of the proton, lenses the light of a star in the Large Magellanic Cloud, which is situated 50 kpc from the Earth. Calculate the Einstein angular radius of the WIMP if it lies at a distance 20 kpc from the Earth. How does this figure compare with the angular radius of the star if it has the same radius, 6.96×10^8 m, as the Sun? Will the microlensing effect of the WIMP be noticeable? Are dark matter microlensing surveys sensitive to the lensing of stars by WIMPs?
What is Einstein angular radius of a brown dwarf with a mass of $0.05 M_\odot$ at the same location as the WIMP? Will the lensing effect of the brown dwarf on the background star be noticeable if there is a suitable alignment?

Appendix D: Example Problems

Problem 25: Question

The separation l between the Galaxy and M31 is expected to obey the equation

$$\frac{\mathrm{d}^2 l}{\mathrm{d}t^2} = -\frac{GM}{l^2} ,$$

over time t, where M is their combined mass, on the assumption that they move only under their mutual gravitational attraction.

Show that, in addition to the parametric solutions involving sin and cos discussed in the lectures, the result $l = (GM\tau_0^2)^{1/3}(\cosh\eta - 1)$, $t = \tau_0(\sinh\eta - \eta)$, where η is a parameter, is also a solution to this equation.

Show also that the power law $l = k\,t^n$ is a third solution to the equation, where k and n are constants, and determine the required values of k and n.

What do these two cases in this question represent physically? What observational constraints show that these solutions are inappropriate to the real Galaxy–M31 system?

Problem 26: Question

The azimuthal velocity of a star in the Galactic disc is v_ϕ and the velocity dispersion in the azimuthal direction is σ_ϕ. If the velocity dispersion at a point is defined by $\sigma_\phi^2 \equiv \langle\, (\, v_\phi - \langle v_\phi \rangle\,)^2\,\rangle$, find an expression for σ_ϕ in terms of $\langle v_\phi^2 \rangle$ and $\langle v_\phi \rangle$.

Obtain similar expressions for σ_R in terms of $\langle v_R^2 \rangle$, and for σ_z in terms of $\langle v_z^2 \rangle$, in a standard (R, ϕ, z) cylindrical coordinate system with $R = 0$ at the Galactic Centre and $z = 0$ in the Galactic plane.

Show that the asymmetric drift v_a, circular velocity v_{circ} and mean value of the azimuthal component $\langle v_\phi \rangle$ of the velocity of stars are related by

$$v_{circ}^2 - \langle v_\phi \rangle^2 = v_a(2v_{circ} - v_a) ,$$

at any point in the Galactic disc.

Argue that the asymmetric drift $v_a \simeq \dfrac{v_{circ}^2 - \langle v_\phi \rangle^2}{2v_{circ}}$ for stars belonging to the Galactic disc.

An analysis based on the Jeans equations in a steady state, axisymmetric potential shows that

$$v_{circ}^2 - \langle v_\phi \rangle^2 = -F\,\langle v_R^2 \rangle ,$$

where F is a factor that depends on the details of the dynamics, but with $F \simeq$ constant. Hence show that the asymmetric drift

$$v_a \simeq -F\,\frac{\sigma_R^2}{2v_{circ}} ,$$

i.e. that $v_a \propto \sigma_R^2$ approximately.