Queen Mary
University of London

M. Sc. Examination by course unit 2010

ASTM002 The Galaxy

Duration: 3 hours

Date and time: 24 May 2010, 2:30pm

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Examiner(s): W.J. Sutherland

Useful information

In this paper π and e represent the conventional mathematical constants.
G is the gravitational constant, with $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}s^{-2}}$.
c is the velocity of light, with $c = 3.00 \times 10^8\ \mathrm{m\,s^{-1}}$.
1 pc $= 3.09 \times 10^{16}$ m.
1 astronomical unit (AU) $= 1.50 \times 10^{11}$ m.
The mass of the Sun is $M_\odot = 2.0 \times 10^{30}$ kg.
The distance of the Sun from the Galactic Centre is $R_0 = 8.0\,\mathrm{kpc}$.

Poisson's equation states that $\nabla^2\Phi = 4\pi G\rho$ at any point in a gravitational field, where Φ is the gravitational potential, G is the constant of gravitation, and ρ is the mass density at that point.

The Laplacian of a scalar function Φ in a spherical coordinate system (r, θ, ϕ) is

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\ .$$

The Laplacian of a scalar function Φ in a cylindrical coordinate system (R, ϕ, z) is

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}\ .$$

The Jeans equations in a steady-state (i.e. time-independent), spherically-symmetric galaxy give the following result

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n\,\langle v_r^2\rangle\right) + \frac{n}{r}\left[\,2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\,\right] = -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\ ,$$

in spherical coordinates, where n is the number density of stars at a distance r from the centre, v_r, v_θ and v_ϕ are the components of the velocity in the r, θ and ϕ directions, and $\Phi(r)$ is the gravitational potential.

In the absence of cosmological effects, the apparent magnitude m of an astronomical object in a photometric band is related to its absolute magnitude M in that band and its distance D from the observer by

$$m - M = 5\log_{10}(D/\mathrm{pc}) - 5 + A\ ,$$

where A is the extinction in the band expressed in magnitudes.

Oort's constants within the Galaxy are defined as

$$A \equiv \frac{1}{2}\left(\frac{\langle v_\phi\rangle}{R} - \frac{\partial\langle v_\phi\rangle}{\partial R}\right) \qquad \text{and} \qquad B \equiv -\frac{1}{2}\left(\frac{\langle v_\phi\rangle}{R} + \frac{\partial\langle v_\phi\rangle}{\partial R}\right)\ ,$$

where $\langle v_\phi\rangle$ is the mean tangential velocity in the Galactic disc, and R is the distance from the Galactic Centre, for $R = R_0$.
[End of Useful Information]

Question 1 (a) Briefly describe the observed properties of spiral galaxies. (You should include reference to their general structure, their stars, gas content, spectra and colours.) [5]

(b) A face-on spiral galaxy is found to have a surface brightness I at distance R from its centre given by $I(R) = I_0 e^{-R/R_0}$, where I_0 and R_0 are constants. What is the total apparent flux from the galaxy in terms of I_0 and R_0 ? [3]

(c) Explain how the Tully-Fisher relation can be used to determine the distances of spiral galaxies. Observations show that two spiral galaxies have rotation velocities respectively $v_{rot} = 150\,\mathrm{km\,s^{-1}}$ and $250\,\mathrm{km\,s^{-1}}$. What is the ratio of their luminosities ? [4]

(d) Explain the difference between dissipational and dissipationless collapse in galaxy formation. Explain why gas is likely to settle to a rotating disc during galaxy formation. [4]

(e) What value does the sum, $A+B$, of the Oort constants have if the rotation curve of the Galaxy is flat near the Sun ? (see definitions in the Useful Information). Express the angular velocity of the Galactic disc at the Sun's distance from the centre in terms of A and B. [4]

Question 2 An isolated system of N stars is bound by their own self-gravity. The ith star has a mass m_i, position vector $\mathbf{x}_i$ and velocity $\dot{\mathbf{x}}_i \equiv d\mathbf{x}_i/dt$ where t is time, and the origin is the centre of mass of the system. The total moment of inertia I of the system is defined as

$$I \equiv \sum_{i=1}^{N} m_i \mathbf{x}_i \cdot \mathbf{x}_i \qquad .$$

(a) Show that

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \sum_{i=1}^{N} m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i + \sum_{i=1}^{N} m_i \mathbf{x}_i \cdot \ddot{\mathbf{x}}_i \qquad .$$

[4]

(b) Give an expression for T, the total kinetic energy of the system. [2]

(c) Give an expression for the gravitational force on star i due to star j in terms of vectors $\mathbf{x}_i$, $\mathbf{x}_j$. Hence write down an expression for the acceleration $\ddot{\mathbf{x}}_i$ of star i as a sum over $j \neq i$. [4]

(d) Hence, prove that

$$\sum_{i=1}^{N} m_i \mathbf{x}_i \cdot \ddot{\mathbf{x}}_i = -\frac{1}{2} \sum_{i,j,(i\neq j)} \frac{G m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \quad ;$$

thus deduce the virial theorem,

$$2\langle T\rangle + \langle U\rangle = 0 \quad ,$$

where U is the total gravitational potential energy. [7]

(e) A cluster of galaxies is observed to have typical galaxy velocities $v \approx 1000\,\mathrm{km\,s^{-1}}$, and a radius $R \approx 500\,\mathrm{kpc}$. Given that the gravitational potential energy U of a uniform-density sphere of mass M and radius R is $U = -3GM^2/5R$, estimate the total mass of the cluster. [3]

Question 3 (a) Explain the meaning of the terms *weak encounter* and *strong encounter* for two stars in a large stellar system approaching each other. [2]

(b) In a weak encounter between two stars of mass m with relative velocity v, the change in the velocity of one star in the reference frame of the other is given by

$$\delta v = \frac{2Gm}{bv} \quad ,$$

where G is the constant of gravitation and b is the impact parameter.

A star moves through a spherical distribution of N stars of overall radius R, with the stars distributed uniformly in space. If the mean change in the square of the velocity is $\delta(v^2) = (\delta v)^2$ in a weak encounter, show that the changes in v^2 caused by all encounters with impact parameters in the range b to $b + \mathrm{d}b$ in a time t is

$$\Delta v^2 = \left(\frac{2Gm}{bv}\right)^2 \left(\frac{3bvtN\mathrm{d}b}{2R^3}\right) \quad .$$

[6]

(c) Hence show that the total change in the square of the velocity in a time t caused by weak encounters with all impact parameters is

$$\Delta v^2(t) = 6\left(\frac{Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \ln\left(\frac{b_{max}}{b_{min}}\right) \quad ,$$

where b_{max} and b_{min} are the largest and smallest values of the impact parameter. [4]

(d) Thereby derive the expression for the relaxation time

$$T_{relax} = \frac{1}{6N\ln\left(\frac{b_{max}}{b_{min}}\right)} \frac{(Rv)^3}{(Gm)^2} \quad .$$

[3]

(e) For a stellar system of radius R containing N stars each of mass m, the typical velocity v is given by $v \approx \sqrt{GNm/R}$. Define the crossing time T_{cross}, and show that for suitable choices of b_{min}, b_{max}, the ratio of the relaxation time to the crossing time is given by

$$\frac{T_{rel}}{T_{cross}} \approx \frac{N}{6\ln N} \quad .$$

[5]

Question 4 (a) The continuity equation for the distribution function f of stars in the six-parameter phase space $(x_1, x_2, x_3, v_1, v_2, v_3)$ of position $\mathbf{x}$ and velocity $\mathbf{v}$ states that

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i} \left(f \frac{\mathrm{d}x_i}{\mathrm{d}t} \right) + \frac{\partial}{\partial v_i} \left(f \frac{\mathrm{d}v_i}{\mathrm{d}t} \right) \right) = 0 \ ,$$

where t is time.
Derive the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\mathrm{d}x_i}{\mathrm{d}t} \frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t} \frac{\partial f}{\partial v_i} \right) = 0$$

from the continuity equation, showing your working. [6]

(b) Derive the first of the Jeans equations,

$$\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial n \langle v_i \rangle}{\partial x_i} = 0 \, ,$$

from the collisionless Boltzmann equation, where n is the number density of stars and $\langle v_i \rangle$ is the mean value of the v_i velocity component at a point. (Explain your working and assumptions). [6]

(c) The velocity dispersion tensor is defined by

$$\sigma_{ij}^2 = \frac{1}{n} \int \Big(v_i - \langle v_i \rangle \Big) \Big(v_j - \langle v_j \rangle \Big) \, f \, \mathrm{d}^3\mathbf{v}$$

where v_1, v_2 and v_3 are the components of the velocity vector $\mathbf{v}$, n is the number density of stars in space, f is the distribution function, and i and $j = 1, 2$ and 3. Prove that

$$\sigma_{ij}^2 = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle \quad .$$

[4]

(d) Explain the meanings of the terms *pressure supported* and *rotationally supported* as applied to a galaxy. Which of these dominate for elliptical and spiral galaxies respectively ? [4]

Question 5 (a) Define the term *integral of motion* as applied to a stellar orbit. For an orbit in any time-independent axisymmetric potential, state *two* integrals of the motion. [3]

(b) Assume that the Galactic halo is spherical, that it has no net rotation, that its velocity dispersion is isotropic and constant, that it has a potential $\Phi(r) = v_0^2 \ln(r/a)$ where v_0, a are constants, and that it has a stellar density profile of the form $n(r) \propto r^{-l}$ where l is a constant. Using a suitable Jeans equation from the Useful Information above, determine an expression for the velocity dispersion σ of halo stars in the solar neighbourhood in terms of v_0 and l.
If $v_0 = 220 \, \mathrm{km \, s^{-1}}$, calculate σ for a plausible value of l. [6]

(c) One of the Jeans equations in a cylindrical coordinate system (R, θ, z) centred on the Galaxy, with $z = 0$ in the plane, gives

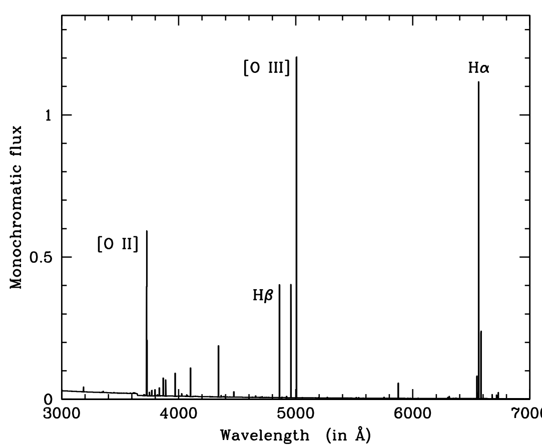
$$\frac{\partial(n\langle v_z\rangle)}{\partial t} + \frac{\partial(n\langle v_R v_z\rangle)}{\partial R} + \frac{\partial(n\langle v_z^2\rangle)}{\partial z} + \frac{n\langle v_R v_z\rangle}{R} = -n\frac{\partial\Phi}{\partial z} \;,$$

where n is the star number density, v_R and v_z are the velocity components in the R and z directions, $\Phi(R, z)$ is the Galactic gravitational potential and t is time. Assuming that the Galaxy is in a steady state, derive an expression for the surface mass density $\Sigma(z, R_0)$ within a distance z of the mid-plane of the Galactic disc at the solar radius R_0 in terms of n and v_z for stars lying towards the Galactic poles. State any other assumptions you make. [8]

(d) Describe the observations of stars necessary in order to apply the above result to measure the surface mass density of the disc of our Galaxy. [3]

Question 6 (a) Briefly describe the character of the dominant electromagnetic radiation emitted by (i) Sun-like stars , and (ii) dust in the intergalactic medium. Give a short explanation of why the typical wavelengths are different. [4]

(b) Explain the meanings of HI, HII, and H_2 referring to hydrogen in the interstellar medium of a galaxy. Which of these gives rise to 21cm emission, and what are the energy levels involved ? [4]



(c) The figure (above) shows the visible-wavelength spectrum of the Orion Nebula. Briefly explain the different emission lines labelled, and the physical mechanisms responsible for producing them. [6]

(d) A star near the Galactic plane is observed to have a Sun-like spectrum, and apparent magnitudes in the blue and visual bands of $m_B = 16.59$, $m_V = 15.54$. Given that the Sun has absolute magnitudes in these bands of $M_B = 5.48$, $M_V = 4.83$, and the reddening ratio for interstellar dust is $A_V/E(B-V) = 3.0$, estimate (i) the reddening of the star, (ii) the V-band extinction in magnitudes, and (iii) the distance to the star. [6]

Question 7 (a) The symbols X, Y and Z denote the fractions by mass of Hydrogen, Helium and heavy elements (metals) respectively, with approximate values of $X \approx 0.71$, $Y \approx 0.27$, $Z \approx 0.02$. Which processes were responsible for creating (i) most of the helium, and (ii) most of the heavy elements ? [2]

(b) List any four assumptions behind the Simple Model of galactic chemical evolution. [4]

(c) In a region of the Galaxy, the total mass of stars is $M_{\rm stars}$, the total mass of interstellar gas is $M_{\rm gas}$, and the mass of heavy elements in the interstellar medium is $M_{\rm metals}$, while the metallicity of the gas is Z. The changes in these quantities in a small time interval are $\delta M_{\rm stars}$, $\delta M_{\rm gas}$, $\delta M_{\rm metals}$ and δZ respectively.
For the Simple Model of galactic chemical enrichment, derive the expression

$$\delta Z = \frac{\delta M_{\rm metals}}{M_{\rm gas}} - Z \frac{\delta M_{\rm gas}}{M_{\rm gas}} .$$

[4]

(d) If $\delta M_{\rm metals}$ and $\delta M_{\rm stars}$ above are related by
$\delta M_{\rm metals} = - Z \delta M_{\rm stars} + p \delta M_{\rm stars}$, where p is the yield of heavy elements, show that

$$\delta Z = - p \frac{\delta M_{gas}}{M_{gas}} .$$

[3]

(e) Hence show that if $n(Z)$ is the number of stars with metallicity less than Z, and Z_1 is the present-day metallicity of the intergalactic medium, the Simple Model predicts that for long-lived stars we should observe

$$\frac{n(Z)}{n(Z_1)} = \frac{1 - e^{-Z/p}}{1 - e^{-Z_1/p}} .$$

[5]

(f) How well does the above prediction match observations for G dwarfs in our galaxy ? [2]

Question 8 (a) The physical size of the Einstein radius for a gravitational lens is

$$r_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}D_L}{D_S}} \quad ,$$

where M is the lens mass, and D_L is the distance from Earth to the lens, D_S is the distance from Earth to the source, and D_{LS} is the distance between lens and source.

Assuming a lens mass $M = 0.1M_\odot$, an Earth–to–source distance $D_S = 50\,\mathrm{kpc}$, and a lens half-way between Earth and source, calculate the Einstein radius. Convert to astronomical units (see Useful Information). [6]

(b) The above lens has a velocity perpendicular to the line of sight of $200\,\mathrm{km\,s^{-1}}$. Estimate the typical duration of a microlensing event which it causes. [2]

(c) Explain why a lens cannot produce a substantial brightening effect by microlensing if r_E/D_L is significantly smaller than the angular radius of the source. For the distances given above, estimate the smallest lens mass which can produce a substantial brightening if the source is a solar-like star of radius $7 \times 10^8\,\mathrm{m}$. [6]

(d) The optical depth τ to microlensing is defined as the mean number of lenses within 1 r_E of the line of sight to a background source star. Show that the optical depth τ through a distribution of microlenses of mass M along a line of sight to a given source is given by

$$\tau \; = \; \frac{4\pi G}{c^2 D_S} \int_0^{D_S} D_L \, D_{LS} \, \rho(D_L) \, \mathrm{d}D_L \qquad ,$$

where $\rho(D_L)$ is the mean mass density of lenses at distance D_L. What is the approximate value of τ along a line of sight through the Galaxy ? [6]

End of Paper