Queen Mary
University of London

M. Sc. Examination by course unit 2009

ASTM002 The Galaxy

Duration: 3 hours

Date and time: 21 May 2009, 1815h–2115h

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 5 questions answered will be counted.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): J.R. Donnison, B.J. Carr

Useful information

In this paper π and e represent the conventional mathematical constants.
G represents the gravitational constant with $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}s^{-2}}$.
$M_\odot$ is the solar mass, with $M_\odot = 1.99 \times 10^{30}$ kg.
c is the velocity of light with $c = 3.00 \times 10^8\ \mathrm{m\,s^{-1}}$.
One year is 3.16×10^7 s.
1 pc $= 3.09 \times 10^{16}$ m.
The distance of the Sun from the Galactic Centre is $R_0 = 8.0$ kpc.

Poisson's equation states that $\nabla^2\Phi = 4\pi G\rho$ at any point in a gravitational field, where Φ is the gravitational potential, G is the constant of gravitation, and ρ is the density at that point.

The Laplacian of a scalar function Φ in a spherical coordinate system (r, θ, ϕ) is

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}.$$

The virial theorem states that the time-averaged total kinetic energy $\langle T\rangle$ of a system of particles in equilibrium is related to the time-averaged total internal potential energy $\langle V\rangle$ by $2\langle T\rangle + \langle V\rangle = 0$.

The following standard integral may be assumed:

$$\int \frac{x^2}{(x^2+a^2)}\,\mathrm{d}x = x - a\tan^{-1}\left(x/a\right) + \text{constant}.$$

Question 1 (a) How does the gas content of galaxies vary with galaxy type? How does this explain the difference in the optical spectra of elliptical and irregular galaxies? [3]

(b) A face-on disc galaxy is found to have a surface brightness I at a distance R from its centre that is given by $I(R) = I_0\,e^{-R/R_0}$, where I_0 and R_0 are constants. What is the total apparent flux from the galaxy in terms of I_0 and R_0?

[6]

(c) A category of galaxies is found to have an observed surface brightness profile $I(R) = I_0\,f(R/R_0)$, where f is some function, and I_0 and R_0 are constants for a given galaxy. Observations show that all the galaxies have the same value I_0 and function f, but different galaxies have different values of R_0.
(i) If the mass-to-light ratio M/L has the same value in all these galaxies, show that

$$L \propto v^4$$

where L is the total luminosity and v is a characteristic velocity. [6]
(ii) How does this relate to the Tully-Fisher relation for spiral galaxies and the Faber-Jackson relation for ellipticals? [2]

(d) A large stationary gas cloud collapses under its own gravitation. Why must this collapse be dissipative? [2]

Question 2 (a) A system of N stars is gravitationally bound. The ith star has a mass m_i, a position vector $\mathbf{x}_i$ and a velocity $\dot{\mathbf{x}}_i \equiv \mathrm{d}\mathbf{x}_i/\mathrm{d}t$ where t is time. A parameter F is defined as $F \equiv \sum_{i=1}^{N} m_i\,\dot{\mathbf{x}}_i.\mathbf{x}_i$.

(i) Show that $\dfrac{\mathrm{d}F}{\mathrm{d}t} = 2T + \sum_{i=1}^{N} m_i\,\ddot{\mathbf{x}}_i.\mathbf{x}_i$, at any time t where T is the total kinetic energy of the system. [4]
(ii) The moment of inertia I is defined as $I \equiv \sum_{i=1}^{N} m_i\,\mathbf{x}_i.\mathbf{x}_i$. Derive a relationship between F and the derivative of I with respect to time. [3]
(iii) By considering the long-term average, show that

$$2\langle T\rangle + \sum_{i=1}^{N} m_i\,\langle \ddot{\mathbf{x}}_i.\mathbf{x}_i\rangle = 0\,. \qquad [4]$$

(iv) Given that $\sum_i m_i\,\ddot{\mathbf{x}}_i.\mathbf{x}_i$ is equal to the potential energy V at time t, derive the virial theorem for a stellar system in dynamical equilibrium. [2]

(b) A spherical galaxy has a uniform density, a total mass M and a radius R. The gravitational potential energy of a uniform sphere of mass M and radius R is given by

$$V = -\frac{3}{5}\frac{GM^2}{R}\,.$$

Using the virial theorem, show how the typical speed v is related to the mass and radius.

[3]

(c) A globular cluster is represented by a Plummer potential with a softening parameter $a = 2\times10^{17}$m. Observations show that the root-mean-square velocity of stars in the cluster is $v = 10\,\mathrm{km\,s^{-1}}$. The total internal potential energy for this potential is

$$V = -\frac{3\pi}{32}\frac{GM_{tot}^2}{a},$$

Estimate the total mass of the cluster in solar masses where M_{tot} is the total mass of the cluster.

[4]

Question 3 (a) (i) Define the crossing time for a system of stars of size D in which the stars have a typical velocity v. [2]

(ii) A spherical system in an equilibrium state has a radius R and consists of N stars of mass m that are evenly distributed. The typical velocity v is related to N, m and R by

$$v^2 \sim \frac{NGm}{R}.$$

Show that the crossing time of this system

$$T_{cross} \sim \frac{1}{\sqrt{G\rho}},$$

where ρ is the mean density of the system. [3]

(b) Define the term *strong encounter* for interactions between two stars of mass m.

[2]

(c) If a star moving with velocity v has a weak encounter with a star of mass m that is at rest, its change in velocity is

$$\delta v \doteq \frac{2Gm}{bv},$$

where b is the impact parameter for the encounter. The star moves through a uniform spherical distribution of N stars of overall radius R.

(i) If the change in the square of the velocity is $\delta v^2 = (\delta v)^2$ in a weak encounter, show that the changes in v^2 caused by all encounters with impact parameters in the range b to $b + db$ in a time t is

$$\Delta v^2 = \left(\frac{2Gm}{bv}\right)^2\left(\frac{3bvtN\,db}{2R^3}\right).$$ [4]

(ii) Hence show that the total change in v^2 in a time t caused by weak encounters with all impact parameters is

$$\Delta v^2(t) = 6\left(\frac{Gm}{v}\right)^2\frac{v\,t\,N}{R^3}\ln\left(\frac{b_{max}}{b_{min}}\right),$$

where b_{max} and b_{min} are the largest and smallest values of the impact parameter. [4]

(d) If the relaxation time T_{relax} is defined as the time taken for the changes in v^2 to reach v^2, i.e. $\Delta v^2(T_{relax}) = v^2$, show that

$$T_{relax} = \frac{1}{6N\ln\left(\frac{b_{max}}{b_{min}}\right)} \frac{(Rv)^3}{(Gm)^2} .$$ [2]

(e) An elliptical galaxy contains 10^{12} stars and has radius 5×10^{20} m. The stars within it move with a typical speed $v = 300\ \mathrm{km\,s^{-1}}$. If the strong encounter radius is 3×10^9 m, estimate the relaxation time for this galaxy. Do encounters between stars have a significant effect on the internal dynamics of this galaxy? [3]

Question 4 (a) A galaxy is modelled using a spherically-symmetric Plummer gravitational potential of the form

$$\Phi(r) = -\frac{GM_{tot}}{\sqrt{r^2 + a^2}}$$

where r is the radial distance from the centre of the galaxy, a and k are constants.

(i) Using Poisson's equation $\nabla^2\Phi = 4\pi G\rho$, show that the mass density ρ as a function of distance r implied by this potential is

$$\rho(r) = \frac{3M_{tot}}{4\pi} \frac{a^2}{(r^2 + a^2)^{5/2}} .$$

[6]

(ii) What is the behaviour of this density as $r \to \infty$? [2]

(iii) Show that the mass $M(r)$ interior to a radius r is

$$M(r) = \frac{M_{tot}\, r^3}{(r^2 + a^2)^{3/2}} .$$

[3]

(iv) How does the circular velocity v_{circ} vary with r for this profile? [2]

(b) The collisionless Boltzmann equation gives

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\mathrm{d}x_i}{\mathrm{d}t} \frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t} \frac{\partial f}{\partial v_i} \right) = 0,$$

where f is the distribution function, t is time, x_i are the components of the position vector $\mathbf{x}$, and v_i are the components of the velocity vector $\mathbf{v}$ for $i = 1, 2, 3$.

Derive from this the second Jeans equation,

$$\frac{\partial(n\langle v_i\rangle)}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(n\langle v_i v_j\rangle) = -\frac{\partial \Phi}{\partial x_i} n ,$$

where $\Phi(\mathrm{x},t)$ is the gravitational potential, and $n(\mathrm{x},t)$ is the number density of stars at a point in space. You may assume that

$$\int v_i \frac{\partial f}{\partial t}\, d^3\mathrm{v} = \frac{\partial}{\partial t}(n\langle v_i\rangle)\,, \qquad \int v_i v_j \frac{\partial f}{\partial x_j}\, d^3\mathrm{v} = \frac{\partial (n\langle v_i v_j\rangle)}{\partial x_j}$$

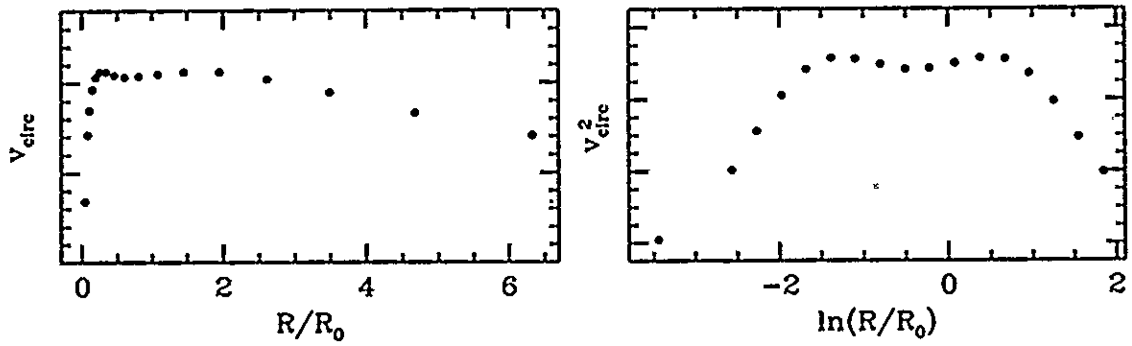
and

$$\int v_i \frac{\partial \Phi}{\partial x_j}\frac{\partial f}{\partial v_j}\, d^3\mathrm{v} = -\frac{\partial \Phi}{\partial x_j}\delta_{ij}\, n\,,$$

where δ_{ij} is the Kronecker delta function. Here $\langle p\rangle$ denotes the mean value of some parameter p. [7]

Question 5 (a) Observations of a spherically symmetric galaxy show that the rotation velocity v_{circ} is constant over radius R. What is the functional form for the variation of the mass $M(R)$ interior to a radius R in terms of radius (i.e. what is $M(R)$ in terms of R)? [3]

(b) The graphs below plot the rotation curve of a spiral galaxy. The first graph shows the circular velocity v_{circ} against R/R_0, where R_0 is the exponential scale length of the disc observed in visible light. The second plots v_{circ}^2 against $\ln(R/R_0)$.



Does the rotation curve of this particular galaxy show signs of a significant dark matter halo? Explain your reasoning. [5]

(c) The dark matter within the Galaxy is sometimes modelled using a spherically-symmetric density distribution

$$\rho(r) = \frac{\rho_0}{1 + r^2/a^2}\,,$$

where $\rho(r)$ is the mass density at a distance r from the centre, and ρ_0 and a are positive constants. (i) Show that the mass interior to a radius r is

$$M(r) = 4\pi\rho_0 a^2\left(r - a\tan^{-1}(r/a)\right). \qquad [6]$$

(ii) Derive an expression for the circular velocity v_{circ} for this mass distribution. What is the dependence of v_{circ} on r at large distances ($r \gg a$)? How does this compare with the observed rotation curve of the Galaxy? [4]
(iii) What is the total mass out to infinity implied by this density profile? How must

the real density profile of the Galaxy behave at large radii compared with this model profile? [2]

Question 6 (a) (i) What is the origin of the 21cm radio emission from the interstellar medium of the Galaxy? [2]
(ii) Why is this line impossible to observe directly in the laboratory? [1]
(b) Is interstellar gas collisional or collisionless? [2]

(c) A star lying near to the Galactic plane is observed to have a visual magnitude of $V = 15.00$ and a blue magnitude of $B = 16.20$. Spectroscopy shows the star to have an intrinsic colour $(B-V)_0 = 0.60$ mag. What is the colour excess of the star? Estimate the extinction in the V band towards the star. [4]

(d) The optical depth caused by dust extinction when light of wavelength λ travels a distance dl through the interstellar medium can be taken to be $d\tau_\lambda = \kappa_\lambda \rho \, dl$, where ρ is the density of dust in space and κ_λ can be taken to be a constant. Assuming that the density of dust varies with distance z above the Galactic plane as $\rho(z) = \rho_0 \, e^{-|z|/h}$, where ρ_0 and h are constants, derive an expression for the total optical depth τ_λ caused by dust extinction along a line of sight through the Galaxy to distant sources as a function of the galactic latitude b. If the extinction in magnitudes, A_λ, is related to the optical depth τ_λ by $A_\lambda = 1.086\,\tau_\lambda$, show that the extinction is given by $A_\lambda = 1.086\,\kappa_\lambda\,\rho_0\,h\,\mathrm{cosec}|b|$. [4]

(e) The separation l between the Galaxy and the Andromeda Galaxy M31 satisfies the equation

$$\frac{d^2 l}{dt^2} = -\frac{GM}{l^2},$$

where M is the combined total mass of the two galaxies and t is time.
Verify that $l = k\,t^n$ is a solution to this equation, where k and n are constants, and determine the required values of k and n.
Discuss how well this solution describes the actual dynamics of M31 and the Galaxy? What dynamical situation does the solution represent? [7]

Question 7 (a) List any four assumptions behind the Simple Model of galactic chemical enrichment. [4]

(b) The change δZ in the heavy element mass fraction Z when the mass of gas $M_{\rm gas}$ in a volume of space changes by $\delta M_{\rm gas}$ is given in the Simple Model by

$$\delta Z = -p\,\frac{\delta M_{\rm gas}}{M_{\rm gas}},$$

where p is the yield of heavy elements. Show that the mass $M_{stars}(t)$ of stars at time t is related to the heavy element fraction $Z(t)$ and the initial gas mass $M_{gas}(0)$ by

$$Z = -p \ln\left(1 - \frac{M_{\rm stars}(t)}{M_{\rm gas}(0)}\right).$$

[5]

(c) Show that in the Simple Model, the mean metallicity of a population of long-lived stars is given by

$$\langle Z \rangle = p\left(\frac{M_{\rm gas}(0)}{M_{\rm stars}} - 1\right)\ln\left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)}\right) + p\ .$$

Show that this can be written in terms of the gas fraction μ as $\langle Z \rangle = p\left(1 + \frac{\mu \ln \mu}{1-\mu}\right)$. What is the mean metallicity $\langle Z \rangle$ when the gas has been used up entirely in star formation? [7]

You may find helpful the standard integral

$$\int \ln(1 - x/a)\, dx = (x-a)\ln(1-x/a) - x + \text{constant.}$$

(d) What are the [O/Fe] parameters for a star with oxygen-to-iron abundance ratios
(i) 1/10
(ii) 1/100
that for the Sun.

[4]

Question 8 (a) The angular size corresponding to the Einstein radius of a gravitational lens is

$$\theta_E = \sqrt{\frac{4GM_L}{c^2}\frac{D_{LS}}{D_L D_S}}\ ,$$

where M_L is the mass of the lens, and D_S, D_L and D_{LS} are the distances from the observer to the light source, from the observer to the lensing object, and between the lens and source, respectively. Show that the optical depth through a distribution of microlenses of mass M_L along a path length to a source is given by

$$\tau = \frac{4\pi G}{c^2 D_S}\int_0^{D_S} D_L\, D_{LS}\, \rho(D_L)\, dD_L\ ,$$

where ρ is the mean mass density of lenses in the relevant volume of space.

[8]

(b) A survey attempts to detect microlensing events from MACHOs by observing a field in the Milky Way at a distance R_f from the Galactic centre in the direction of the Galactic anticentre.

(i) Assuming that the dark matter halo of the Galaxy can be represented by an isothermal sphere of compact objects with a density distribution

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2} ,$$

where r is the radial distance from the Galactic centre and σ is the velocity dispersion of the compact objects (a constant in space), show that the optical depth for microlensing the star field is

$$\tau = \frac{2\sigma^2}{c^2} \left[\left(\frac{R_f + R_0}{R_f - R_0} \right) \ln \left(\frac{R_f}{R_0} \right) - 2 \right] ,$$

where R_0 is the distance of the Sun from the Galactic Centre. You may assume that

$$\int \frac{x}{(a+x)^2}\, dx = \ln \left| a + x \right| + \frac{a}{a+x} + \text{constant}$$

and

$$\int \frac{x^2}{(a+x)^2}\, dx = x - 2a \ln \left| a + x \right| - \frac{a^2}{a+x} + \text{constant} .$$

[8]

(ii) If $R_f = 2R_0$ and $\sigma = 200\,\mathrm{km\,s^{-1}}$, estimate τ to within an order of magnitude. What does this imply for the number of stars that would have to be studied in the microlensing survey? [4]

End of Paper