(b) The dark matter within our Galaxy is sometimes modelled using a spherically-symmetric density distribution

$$\rho(r) = \frac{\rho_0}{1 + r^2/a^2} ,$$

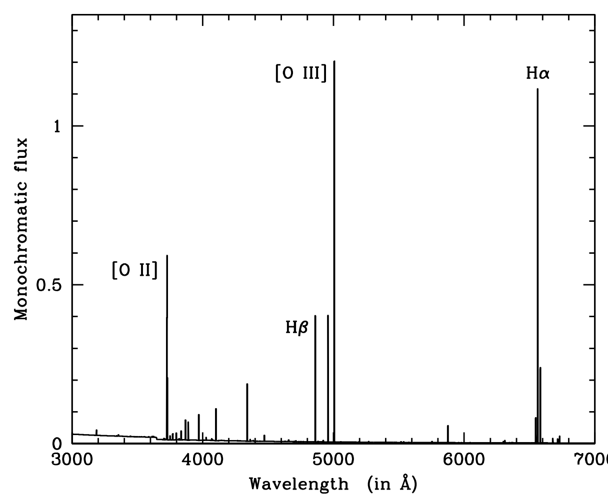
where $\rho(r)$ is the mass density at a distance r from the centre, and ρ_0 and a are positive constants. Show that the mass interior to a radius r is

$$M(r) = 4\pi\rho_0\, a^2 \left(r - a\tan^{-1}(r/a) \right) . \qquad [6]$$

Derive an expression for the circular velocity v_{circ} for this mass distribution. What is the dependence of v_{circ} on radial distance r at large distances ($r \gg a$)? How does this compare with the observed rotation curve of the Galaxy? [4]
What is the total mass out to infinity implied by this density profile? How must the real density profile of the Galaxy behave at large radii compared with this model profile? [2]

[Total 20 marks for question]

6. (a) The figure (right) shows the optical spectrum of the Orion Nebula. Briefly explain the different emission lines and the physical mechanisms responsible for producing them. [4 marks]



(b) What physical mechanism is responsible for maser emission of radiation from the interstellar medium in the immediate vicinity of some radiation sources? [2]

(c) Is gas collisional or collisionless in an astronomical context? [2]

(d) A star lying in the Galactic plane is observed to have a visual magnitude of $V = 17.15$ and a blue magnitude of $B = 18.45$. Spectroscopy shows the star to be of a type that has an intrinsic colour $(B-V)_0 = 0.80$ mag and a V-band absolute magnitude $M_V = +5.60$. What is the colour excess of the star? Estimate the extinction in the V band towards the star. Estimate the distance of the star from the Sun.
If the star has a Galactic longitude of $l = 180°$, estimate its distance from the Galactic Centre. [5]

[This question continues overleaf . . .]

(e) The mean density in the form of stars in the disc of the Galaxy is observed to vary with the distance z from the Galactic plane as $\rho_s(z) = \rho_{so}\mathrm{e}^{-|z|/h_s}$ close to the Sun, where ρ_{so} is the density of stars in the plane, and h_s is a scale height. The density of the interstellar gas ρ_g is also found to vary roughly exponentially with height with $\rho_g(z) = \rho_{go}\mathrm{e}^{-|z|/h_g}$, where ρ_{go} and h_g are constants. Observations show that $h_s = 250\,\mathrm{pc}$, $h_g = 150\,\mathrm{pc}$ and $\rho_{so} = 6\,\rho_{go}$.

What is the ratio of the surface density of stars, Σ_s, to that of gas, Σ_g, at the Sun's distance from the Galactic Centre? [4]

How do you expect the surface density of the dust, Σ_d, to compare with Σ_s? [1]

How realistic is this representation of the gas density as $\rho_g(z) = \rho_{go}\mathrm{e}^{-|z|/h_g}$ in practice? [2]

[Total 20 marks for question]

7. (a) List any four assumptions behind the Simple Model of galactic chemical evolution. [4 marks]

(b) In a region of the Galaxy, the total mass of stars is M_{stars}, the total mass of interstellar gas is M_{gas}, and the mass of heavy elements in the interstellar medium is M_{metals}, while the metallicity of the gas is Z. The changes in these quantities in a small time interval are $\delta M_{\mathrm{stars}}$, δM_{gas}, $\delta M_{\mathrm{metals}}$ and δZ respectively.
For the Simple Model of galactic chemical enrichment derive the expression

$$\delta Z = \frac{\delta M_{\mathrm{metals}}}{M_{\mathrm{gas}}} - Z\,\frac{\delta M_{\mathrm{gas}}}{M_{\mathrm{gas}}} . \qquad [4]$$

If $\delta M_{\mathrm{metals}}$ and $\delta M_{\mathrm{stars}}$ are related by $\delta M_{\mathrm{metals}} = -\,Z\,\delta M_{\mathrm{stars}} + p\,\delta M_{\mathrm{stars}}$, where p is the yield of heavy elements, show that

$$\delta Z = -\,p\,\frac{\delta M_{gas}}{M_{gas}} . \qquad [3]$$

Hence show that the metallicity Z of the interstellar medium is related to the gas fraction μ by

$$Z = -\,p\ln\mu . \qquad [4]$$

How well does this prediction match observations of the gas in galaxies? [3]

(c) A Galactic star is observed to have [Fe/H] = −2.0. To which component of the Galaxy is it likely to belong? How old is it likely to be? [2]

[Total 20 marks for question]

[Next question overleaf]

8. (a) The angular size corresponding to the Einstein radius of a gravitational lens is

$$\theta_E = \sqrt{\frac{4GM_L}{c^2}\frac{D_{LS}}{D_L D_S}} \quad ,$$

where M_L is the mass of the lens, and D_S, D_L and D_{LS} are the distances from the observer to the light source, from the observer to the lensing object, and from the lens to the source respectively. Show that the optical depth through a distribution of microlenses of mass M_L along a path length to a source is given by

$$\tau = \frac{4\pi G}{c^2 D_S}\int_0^{D_S} D_L\, D_{LS}\, \rho(D_L)\, \mathrm{d}D_L \quad ,$$

where ρ is the mean mass density of lenses. [6 marks]

(b) A survey attempts to detect microlensing events from MACHOs by observing a field in the Galactic Bulge close to the Galactic Centre. Assume that the dark matter halo is made from compact objects (MACHOs) with approximately stellar mass and with a density distribution

$$\rho(r) = \frac{\rho_0\, a^2}{r^2 + a^2} \quad ,$$

where r is the radial distance from the Galactic Centre, ρ_0 is the central dark matter density and a is a constant. Show therefore that the optical depth of microlensing to the field is

$$\tau = \frac{2\pi G\rho_0 a^2}{c^2}\left(\ln\left(1 + \frac{R_0^2}{a^2}\right) + \frac{2a}{R_0}\tan^{-1}\frac{R_0}{a} - 2\right) ,$$

where R_0 is the distance of the Sun from the Galactic Centre. You may assume that the star field is not significantly affected by dust extinction for this calculation. You may find helpful the standard integral

$$\int \frac{x\,(b-x)}{(b-x)^2 + a^2}\,\mathrm{d}x = -x - a\tan^{-1}\left(\frac{b-x}{a}\right) - \frac{1}{2}b\ln\left(a^2 + (b-x)^2\right) + \text{constant}.$$

[8]

Estimate τ to within an order of magnitude if $R_0 = 8.0$ kpc, $a = 2.0$ kpc and $\rho_0 = 2.0\times10^{-20}\,\mathrm{kg\,m^{-3}}$. What does this imply for the number of stars that would have to be studied in the microlensing survey? [3]

(c) Draw a simple sketch illustrating how the brightness of a distant star varies with time before, during and after a microlensing event. [3]

[Total 20 marks for question]

[End of examination paper]