Appendix D: Example Problems

Problem 8: Answer

The density depends on radius r and does not depend on angle: the profile is spherically symmetric. Therefore the equation of continuity of mass $\mathrm{d}M/\mathrm{d}r = 4\pi r^2\rho$ applies. Substituting for the profile ρ,

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \, \frac{k}{r(r+a)^2} \; .$$

Integrating from the centre to radius r,

$$\int_0^{M(r)} \mathrm{d}M = \int_0^r 4\pi r'^2 \, \frac{k}{r'(r'+a)^2} \, \mathrm{d}r' \qquad \therefore \; M(r) = 4\pi k \int_0^r \frac{r'}{(r'+a)^2} \, \mathrm{d}r' \; .$$

This integral can be solved using

$$\int \frac{r}{(r+a)^2} \, \mathrm{d}r = \int \frac{r+a-a}{(r+a)^2} \, \mathrm{d}r = \int \frac{r+a}{(r+a)^2} \, \mathrm{d}r - a \int \frac{1}{(r+a)^2} \, \mathrm{d}r$$

$$= \int \frac{1}{(r+a)} \, \mathrm{d}r + a \, \frac{1}{(r+a)} = \ln(r+a) + \frac{a}{(r+a)} + c \; .$$

$$\therefore \; M(r) = 4\pi k \left[\ln(r'+a) + \frac{a}{(r'+a)} \right]_0^r$$

$$= 4\pi k \left(\ln(r+a) + \frac{a}{(r+a)} - \ln(a) + \frac{a}{(a)} \right)$$

$$\text{So} \qquad M(r) = 4\pi k \left(\ln\left(\frac{r}{a}+1\right) - \frac{r}{(r+a)} \right) \; ,$$

which is the mass inside a radius r that the question asks for.
As $r \to \infty$, $M(r) \to \infty$. So the model is not physically realistic at large radii.

For spherical symmetry, $\dfrac{GM(r)}{r^2} = \dfrac{\mathrm{d}\Phi}{\mathrm{d}r}$. Substituting for $M(r)$ and integrating from infinity to radius r,

$$\int_0^{\Phi(r)} \mathrm{d}\Phi = \int_\infty^r \frac{4\pi Gk}{r^2} \left(\ln\left(\frac{r}{a}+1\right) - \frac{r}{(r+a)} \right) \mathrm{d}r$$

because the potential is 0 at $r \to \infty$. Therefore,

$$\Phi(r) - 0 = 4\pi Gk \int_\infty^r \left(\frac{1}{r^2} \ln\left(\frac{r}{a}+1\right) - \frac{1}{r(r+a)} \right) \mathrm{d}r$$

The integrals can be solved fairly easily.

$$\int \frac{1}{r(r+a)} \, \mathrm{d}r = \frac{1}{a} \int \left(\frac{1}{r} - \frac{1}{r+a} \right) \mathrm{d}r \quad \text{using partial fractions}$$

$$= \frac{1}{a} \Big(\ln r - \ln(r+a) \Big) + c = \frac{1}{a} \ln\left(\frac{r}{r+a} \right) + c$$

Using integration by parts,

$$\int \frac{1}{r^2} \ln\left(\frac{r+a}{a}\right) \, dr \;=\; -\frac{1}{r}\ln\left(\frac{r+a}{a}\right) \;+\; \int \frac{1}{r}\,\frac{1}{r+a}\, dr$$

$$= -\frac{1}{r}\ln\left(\frac{r+a}{a}\right) + \frac{1}{a}\ln\left(\frac{r}{r+a}\right) + c \qquad \text{using the integral above.}$$

Using these integrals, we get for the potential

$$\Phi(r) \;=\; 4\pi Gk \left[-\frac{1}{r}\ln\left(\frac{r+a}{a}\right) \;+\; \frac{1}{a}\ln\left(\frac{r}{r+a}\right) \;-\; \frac{1}{a}\ln\left(\frac{r}{r+a}\right) \right]_{\infty}^{r}$$

$$= \; 4\pi Gk \left[-\frac{1}{r}\ln\left(\frac{r+a}{a}\right) \right]_{\infty}^{r} \;=\; 4\pi Gk \left(-\frac{1}{r}\ln\left(\frac{r+a}{a}\right) \;-\; 0 \right)$$

So the potential at a distance r from the centre is

$$\Phi(r) \;=\; -\,\frac{4\pi Gk}{r} \ln\left(\frac{r}{a} + 1\right) \;.$$

The central density is infinite. This appears not to be physically realistic. (However, one of the people who first used this profile to describe galaxies – Carlos Frenk – has argued that the profile might be realistic in the cores of galaxies after all. He was thinking about the possibility that most massive galaxies have black holes – and therefore a singularity – at their cores.)

Problem 9: Answer

Assume that the stars are distributed with uniform density across the cluster and that it is spherical with a radius $R = 70$ pc $= 70 \times 3.0857 \times 10^{16}$ m $= 2.2 \times 10^{18}$ m. The typical velocity of the stars is $100\,\mathrm{km\,s^{-1}} = 10^5\,\mathrm{m\,s^{-1}}$. The relaxation time is given by

$$T_{relax} \;\simeq\; \frac{1}{6N\ln\left(\frac{b_{max}}{b_{min}}\right)}\,\frac{(Rv)^3}{(Gm)^2} \;,$$

for a uniform spherical system of radius R containing N stars of mass m moving with velocity v, where b_{max} and b_{min} are the maximum and minimum values of the impact parameter for stellar encounters and G is the constant of gravitation.
To proceed, make the approximation that all stars are moving with the typical velocity ($v = 10^5\,\mathrm{m\,s^{-1}}$), that all stars have a mass of one solar mass ($m = 1M_\odot = 1.989 \times 10^{30}$ kg), and that the maximum impact parameter is the strong encounter radius, $r_S = 2Gm/v^2$. So,

$$T_{relax} \;\simeq\; \frac{1}{6N\ln\left(\frac{Rv^2}{2Gm}\right)}\,\frac{(Rv)^3}{(Gm)^2}$$

$$\simeq\; \frac{1}{6\times 10^7 \ln\left(\frac{2.2\times10^{18}\times(10^5)^2}{2\times6.673\times10^{-11}\times1.989\times10^{30}}\right)}\,\frac{(2.2\times10^{18}\times10^5)^3}{(6.673\times10^{-11}\times1.989\times10^{30})^2}\ \mathrm{s}$$

$$\simeq\; \frac{1}{6\times10^7\ln(8.29\times10^6)}\,\frac{1.06\times10^{70}}{(1.76\times10^{40}}\ \mathrm{s} \;=\; 5.50\times10^{20}\ \mathrm{s}$$

$$\simeq\; 5.50\times10^{20}/3.1557\times10^7\ \mathrm{yr} \;=\; 1.7\times10^{13}\ \mathrm{yr}$$

The relaxation time for this cluster around the nucleus is much longer than the likely age of the galaxy (age of galaxy $\simeq$ age of Universe $= 13.7 \times 10^9$ yr). So the dynamics of the stars can be modelled as a collisionless system over the lifetime of the galaxy. The stars away from the nuclear region of the galaxy are also collisionless: in this case, the stars around the nucleus behave in the same way as the general stars of the galaxy as far as dynamical collisions are concerned.

[It should be noted that the densities of stars around the nuclei of many galaxies are so large that their dynamics are collisional: the relaxation times are small compared to the age of the galaxies. The case given in this problem is an exception.]

Problem 10: Answer

The potential is triaxial. This is apparent from the complexity of the orbit. The orbit maps out a three-dimensional volume in space. There is no obvious "rosette" pattern, either in a static plane (as is found in a spherically-symmetric potential), or in a plane that precesses over time (as in an oblate/prolate potential).

Stars in a triaxial potential map out a volume in space, with the volume defined by the energy and angular momentum of the star. In general the orbits are chaotic.

Problem 11: Answer

Try $f = b\,(-E_m)^{7/2}$, where E_m is the energy per unit mass and b is a constant. The density becomes

$$\rho(r) \;=\; 4\pi\,\sqrt{2}\,\overline{m}\,b \int_{\Phi}^{0} \sqrt{E_m - \Phi}\;(-E_m)^{7/2}\,\mathrm{d}E_m \;\;.$$

Note that E_m and Φ are both negative, so $-E_m$ and $-\Phi$ are both positive. Use the substitution $E_m = \Phi\cos^2\theta$. Differentiating, $\mathrm{d}E_m = -2\Phi\sin\theta\cos\theta\,\mathrm{d}\theta$. The limits of the integral are:

$$\begin{aligned} \text{when}\quad E_m = &\;\Phi,\;\; \cos^2\theta = 1\;. \quad \therefore \cos\theta = \pm 1\;. \quad \text{Take } \theta = 0\;\;. \\ \text{when}\quad E_m = &\;0,\;\; \cos^2\theta = 0\;. \quad \therefore \cos\theta = 0\;. \quad \text{Take } \theta = \pi/2\;\;. \end{aligned}$$

Using this substitution, the density becomes

$$\begin{aligned} \rho(r) &= 4\pi\,\sqrt{2}\,\overline{m}\,b \int_0^{\pi/2} \sqrt{\Phi\cos^2\theta - \Phi}\;(-\Phi\cos^2\theta)^{7/2}\;(-2\Phi\sin\theta\cos\theta\mathrm{d}\theta) \\ &= 4\pi\,\sqrt{2}\,\overline{m}\,b \int_0^{\pi/2} \sqrt{-\Phi}\;\sin\theta\;(-\Phi)^{7/2}\;\cos^7\theta\;(2)(-\Phi)\;\sin\theta\;\cos\theta\;\mathrm{d}\theta \\ &= 8\pi\,\sqrt{2}\,\overline{m}\,b\,(-\Phi)^5 \int_0^{\pi/2} \sin^2\theta\;\cos^8\theta\;\mathrm{d}\theta\;\;. \end{aligned}$$

Using the standard integral, we get,

$$\rho(r) \;=\; 8\pi\,\sqrt{2}\,\overline{m}\,b\,(-\Phi)^5 \left(\frac{7\pi}{512}\right) \;=\; \frac{7\,\sqrt{2}\,\pi^2}{64}\,\overline{m}\,b\,(-\Phi)^5$$

Substituting for the Plummer potential $\Phi(r) = -GM_{tot}/\sqrt{r^2+a^2}$ from Question 4, we obtain,

$$\rho(r) \;=\; \frac{7\,\sqrt{2}\,\pi^2}{64}\,\overline{m}\,b\,\frac{G^5\,M_{tot}^5}{(r^2+a^2)^{5/2}} \;\;.$$

This is the same as the expression for the density of the Plummer potential, $\rho(r) = 3M_{tot}\,a^2/4\pi(r^2+a^2)^{5/2}$, in Question 4 if

$$b \;=\; \frac{24\sqrt{2}}{7\,\pi^3}\,\frac{a^2}{\overline{m}\,G^5\,M_{tot}^4} \;\;.$$

So $f(-E_m) = b(-E_m)^{7/2}$ is a solution to the equation relating density and the distribution function in the question if the constant b has this value.
This is the proof the question asked for.

The energy per unit mass for the position and velocity in the question is

$$E_m = -\frac{G\,M_{tot}}{\sqrt{r^2+a^2}} + \frac{1}{2}v^2 = -8.48\times10^{11} + 2.00\times10^{10} \;\; \mathrm{J\,kg^{-1}} = -8.28\times10^{11} \;\; \mathrm{J\,kg^{-1}}$$

using a radial distance from the centre of the galaxy of $r = \sqrt{x^2+y^2+z^2} = 10\,\mathrm{kpc} = 3.09\times10^{20}$ m. But $f = b\,(-E_m)^{7/2}$ with

$$\begin{aligned}
b &= \frac{24\sqrt{2}}{7\,\pi^3}\,\frac{a^2}{\overline{m}\,G^5\,M_{tot}^4} \\
&= 0.1564 \times \frac{(1.70\times3.0857\times10^{19})^2}{(0.70\times1.989\times10^{30})(6.673\times10^{-11})^5\,(2\times10^{12}\times1.989\times10^{30})^4} \;\; \mathrm{m^{-13}\,s^{10}} \\
&= 0.1564 \times \frac{(1.70\times3.0857)^2\times10^{38}}{(0.70\times1.989)(6.673)^5\,(2\times1.989)^4\times10^{143}} \;\; \mathrm{m^{-13}\,s^{10}} \\
&= 9.33\times10^{-112} \;\; \mathrm{m^{-13}\,s^{10}}
\end{aligned}$$

So $f = b\,(-E_m)^{7/2}$ gives,

$$\begin{aligned}
f &= 9.33\times10^{-112}\,(8.28\times10^{11})^{7/2} \;\; \mathrm{m^{-3}\,(m\,s^{-1})^{-3}} \\
&= 4.82\times10^{-70} \;\; \mathrm{m^{-3}\,(m\,s^{-1})^{-3}} \;\; .
\end{aligned}$$

This means that the density of stars in the six-dimensional phase space is $4.82\times10^{-70}\;\;\mathrm{m^{-3}}$ stars $\mathrm{(m\,s^{-1})^{-3}}$

[Part of this question appeared in the May 2005 examination.]

Appendix D: Example Problems

Problem 12: Answer

The first part of the question is another density–mass calculation for a spherically-symmetric galaxy. Because we have spherical symmetry, for a thin spherical shell we have, $\mathrm{d}M(r) = 4\pi r^2 \rho(r)\,\mathrm{d}r$. Integrating from the centre ($r = 0$) to a radial distance r, we obtain,

$$M(r) = 4\pi\rho_0 \int_0^r \frac{r'^2}{1 + r'^2/a^2}\,\mathrm{d}r' = 4\pi\rho_0\, a^2 \left(r - a\tan^{-1}(r/a) \right)$$

(using a substitution $r' = a\tan\theta$ to solve the integral).
When $r \ll a$, the standard expansion $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + \ldots$ (for $|x| \leq 1$) gives

$$M(r) = 4\pi\rho_0\, a^2 \left(r - r + \frac{r^3}{3a^2} - \mathrm{O}(r^5) \right) = \frac{4\pi\rho_0\, r^3}{3} - \mathrm{O}(r^5)\ .$$

Therefore, $M(r) \propto r^3$ when $r \ll a$.
[This is what we would expect from the fact that $\rho \simeq \rho_0$, a constant, for $r \ll a$, i.e. near the centre. Therefore, near the centre, $M(r) \simeq \rho \times$ volume of sphere of radius r, which gives $M(r) \simeq 4\pi\rho_0 r^3/3$.]
When $r \gg a$, $r/a \gg 1$, which gives $\tan^{-1}(r/a) \simeq \tan^{-1}(1) = \pi/2$, and therefore $r - a\tan^{-1}(r/a) \simeq r - a\pi/2 \simeq r$, which gives, $M(r) = 4\pi\rho_0\, a^2\, r$. Therefore, $M(r) \propto r$ when $r \gg a$.
[This is an important result because $M(r) \propto r$ gives a circular velocity v_{circ} =constant, as is observed in the rotation curves of spiral galaxies.]
Actually the asymptotic forms are obvious from $\rho \sim \rho_0$ (small-r) and $\rho \sim \rho_0/r^2$ (large-r).

The second part of the question can be solved using the Jeans equations expressed in a spherical coordinate system (r, θ, ϕ) centred on the galaxy. For a spherically-symmetric potential, the second Jeans equation gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(n_p\, \langle v_r^2 \rangle \Big) + \frac{n_p}{r} \Big[2\langle v_r^2 \rangle - \langle v_\theta^2 \rangle - \langle v_\phi^2 \rangle \Big] = -\, n_p\, \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

(given in Section **??** of the course notes), where n_p is the number density of some system of particles or stars, Φ is the gravitational potential, while $\langle v_r^2 \rangle, \langle v_\theta^2 \rangle$ and $\langle v_\phi^2 \rangle$ are the mean values of the squares of the velocity components of the particles in the r, θ and ϕ directions. Here we use n_p for the subpopulation of particles, rather than the total number density of all stars (but Φ is the total gravitational potential).
For an isotropic distribution with no net rotation, $\langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle = \sigma^2$, where σ is a constant (and a scalar). So,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n_p\, \sigma^2 \right) + 0 = -\, n_p\, \frac{\partial\Phi}{\partial r}\ .$$

To find $\mathrm{d}\Phi/\mathrm{d}r$, use the fact that the acceleration due to gravity is $\mathbf{g} = -\,\boldsymbol{\nabla}\Phi$ and that $g = GM(r)/r^2$ for a spherical distribution of mass, giving $\mathrm{d}\Phi/\mathrm{d}r = GM(r)/r^2$.

Since the motions of the particles are isothermal, the velocity dispersion is the same everywhere, and so σ is independent of r. Therefore,

$$\sigma^2 \frac{\mathrm{d}n_p}{\mathrm{d}r} = -\, n_p \frac{GM(r)}{r^2}$$

in terms of $M(r)$ and r. Using $\rho_p = \overline{m}_p\, n_p$ where $\overline{m}_p$ is the mean mass of each particle,

$$\sigma^2 \frac{1}{\overline{m}_p} \frac{\mathrm{d}\rho_p}{\mathrm{d}r} = -\, \frac{\rho_p}{\overline{m}_p} \frac{GM(r)}{r^2} \qquad \therefore \quad \sigma^2 \frac{\mathrm{d}\rho_p}{\mathrm{d}r} = -\, \rho_p \frac{GM(r)}{r^2} \; .$$

Integrating,

$$\sigma^2 \int \frac{\mathrm{d}\rho_p}{\rho_p} = -\, G \int \frac{M(r)}{r^2}\, \mathrm{d}r \; ,$$

which gives,

$$\ln \rho_p = -\, \frac{G}{\sigma^2} \int \frac{M(r)}{r^2}\, \mathrm{d}r \; .$$

This is the density of the test particles as a function of r and $M(r)$ asked for in the question (the question does not ask us to go any further: we do not need to integrate this directly here).

For large r, we have $M(r) = 4\pi\rho_0\, a^2\, r$, from the answer to the first part of the question. We can use this large-r approximation in the expression for $\ln \rho_p$ in terms of $M(r)$ above. This will enable us to calculate ρ_p by integrating, for the case where r is large. Doing this gives,

$$\ln \rho_p = -\, \frac{4\pi G \rho_0\, a^2}{\sigma^2} \int \frac{\mathrm{d}r}{r} = -\, \frac{4\pi G \rho_0\, a^2}{\sigma^2} \ln r \, + \, k_1 \; ,$$

for large r, where k_1 is a constant of integration. Rearranging,

$$\rho_p = k_2\, r^{-\frac{4\pi G \rho_0\, a^2}{\sigma^2}} \; ,$$

for large r, where k_2 is a constant. This is a power law of the form $\rho_p = r^{-l}$ where the index is $l = 4\pi G \rho_0\, a^2/\sigma^2$. So the density of the population of test particles will have a power law dependence on distance r at large radii.
Here $\sqrt{l}$ can be interpreted as the ratio of circular speed to dispersion. The tracer population will have the same large-r density law as the massive population if $l = 2$ (i.e. both $\rho \propto r^{-2}$ and $\rho_p \propto r^{-2}$ at large r if $l = 2$). Otherwise the tracer population does not follow the same density distribution as the main population (because they have different velocity charactersitics).

Problem 13: Answer

The crossing time of the globular cluster will be $T_{cross} \sim 20\,\mathrm{pc}/15\,\mathrm{km\,s^{-1}} \sim 1.3\times 10^6\,\mathrm{yr}$, while that of the galaxy will be $\sim 20\,\mathrm{kpc}/200\,\mathrm{km\,s^{-1}} \sim 1.0 \times 10^8\,\mathrm{yr}$. So, the globular cluster will be $\sim 10^4 T_{cross}$ old, whereas the galaxy will be $\sim 10^2 T_{cross}$ old. N-body modelling of the dynamics of the two objects will use a series of time steps, with the positions of the particles being computed at each of these steps. However, the globular cluster simulation will need steps $\sim 10^2$ times smaller than for the galaxy to achieve steps representing the same fraction of the crossing time in both systems. Using $T_{relax}/T_{cross} \simeq N/12 \ln N$, the globular cluster will have $T_{relax} \simeq 700\, T_{cross} \sim 10^9$ yr, so the globular cluster will be $\sim 10\, T_{relax}$ old. In contrast, the galaxy will be $\ll T_{relax}$ old. The globular cluster simulation will therefore have to consider two-body relaxation, while the galaxy simulation can ignore it. Both these considerations make the globular cluster simulation more difficult.