ASTM002 / MAS430 The Galaxy

May 2008 Examination Paper Model Answers

1. (a) Elliptical galaxies have smooth, generally featureless morphologies, and have elliptical (or circular) shapes. They have large populations of globular clusters. They are very gas-poor, show no star formation and have mostly old stars, dominated by spectral type K. Their colours are moderately red, indicative of evolved stellar populations. Their spectra show composite stellar absorption-line spectra, with no interstellar emission lines. [Lectures] [5]

(b) The luminosity of a galaxy $L \propto I_0 R_S^2$, because I_0 represents the light emitted from unit area. Rearranging the expression for the fundamental plane given in the information section, the scale size $R_s \propto I_0^{-0.8}\sigma_0^{1.3}$. Substituting this into the expression for the luminosity, $L \propto I_0(I_0^{-0.8}\sigma_0^{1.3})^2 \propto I_0^{-0.6}\sigma_0^{2.6}$. Hence $L \propto \sigma_0^{2.6}/I_0^{0.6}$.
[Unseen application of principles from lectures] [3]

(c) The presence of emission lines implies that the galaxy is either a spiral or irregular, while the presence of a strong continuum indicates that the emission lines are not as strong as in an irregular. Therefore the galaxy is a spiral. [Lectures] [2]

(d) Mechanical energy is lost during the collapse (being converted into heat): the collapse is dissipative. [Lectures and coursework] [2]

(e) From the definitions of A and B given in the information section,

$$A + B = \frac{1}{2}\left(\frac{\langle v_\phi \rangle}{R} - \frac{\partial \langle v_\phi \rangle}{\partial R}\right) - \frac{1}{2}\left(\frac{\langle v_\phi \rangle}{R} + \frac{\partial \langle v_\phi \rangle}{\partial R}\right) = -\frac{\partial \langle v_\phi \rangle}{\partial R}$$

for $R \simeq R_0$. For a flat rotation curve, $\langle v_\phi \rangle =$ constant, hence $A + B = 0$.
[Lectures] [2]

The angular velocity of the Galactic disc is $\Omega(R) = \langle v_\phi \rangle / R$. From the definition of Oort's constants we find $\langle v_\phi \rangle / R = A - B$. Hence the angular velocity is $\Omega = A - B$.
[Lectures] [1]

(f) Asymmetric drift is the lag in the azimuthal component of the mean velocity of a subpopulation of stars relative to the local standard of rest. [Lectures] [2]
Among the stars of the Galactic disc, there is a gradual increase in asymmetric drift with age. The velocity dispersion of disc stars increases with age. Encounters with giant molecular clouds will have perturbed the motions of old stars, increasing their velocity dispersions. [Lectures] [3]

[Total 20 marks for question]

2. (a) Consider a star of mass m approaching a perturbing star of mass m with an impact parameter b. Because the encounter is weak, the change in the direction of motion will be small and the change in velocity will be perpendicular to the initial direction of motion. At any time t when the separation is r, the component of the gravitational force perpendicular to the initial direction of motion will be

$$F_{perp} = \frac{Gm^2}{r^2}\cos\phi \quad ,$$

where ϕ is the angle at the perturbing mass between the point of closest approach and the perturbed star. [2]

Using $\cos\phi = b/r$ and making the approximation that the speed along the trajectory is constant, $r \simeq \sqrt{b^2 + v^2 t^2}$ where t is time and $t = 0$ at the point of closest approach. By applying $F = ma$ perpendicular to the initial direction of motion we obtain

$$\frac{\mathrm{d}v_{perp}}{\mathrm{d}t} = \frac{Gm}{r^2}\cos\phi = \frac{G\,m\,b}{(b^2+v^2t^2)^{3/2}} \quad ,$$

where v_{perp} is the component at time t of the velocity perpendicular to the intial direction of motion. [Lectures] [3]

Integrating from time $-\infty$ to ∞,

$$v_{perp} = G\,m\,b\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(b^2+v^2t^2)^{3/2}} \quad .$$

Using the standard integral in the question, we obtain the final formula.

[Lectures] [3]

Consider all weak encounters occurring in a time period t that have impact parameters in the range b to $b+\mathrm{d}b$ within a uniform spherical system of N stars and radius R. The volume swept out by impact parameters b to $b+\mathrm{d}b$ in time t is $2\,\pi\,b\,\mathrm{d}b\,v\,t$. Therefore the number of stars encountered with impact parameters between b and $b+\mathrm{d}b$ in time t is (volume swept out) (number density of stars)

$$= \left(2\,\pi\,b\,\mathrm{d}b\,v\,t\right)\frac{N}{\frac{4}{3}\pi R^3} = \frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2R^3} \quad .$$

[Lectures] [2 marks]

The total change in v^2 caused by all encounters in time t with impact parameters in the range b to $b+\mathrm{d}b$ will be

$$\Delta v^2 = \left(\frac{2Gm}{bv}\right)^2\left(\frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2\,R^3}\right) \quad .$$

[Lectures] [2 marks]

Integrating over b, the total change in a time t from all impact parameters from b_{min} to b_{max} is

$$\Delta v^2(t) = \int_{b_{min}}^{b_{max}}\left(\frac{2Gm}{bv}\right)^2\left(\frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2\,R^3}\right) = \frac{3}{2}\left(\frac{2Gm}{v}\right)^2\frac{v\,t\,N}{R^3}\int_{b_{min}}^{b_{max}}\frac{\mathrm{d}b}{b}$$

$$\therefore \quad \Delta v^2(t) = 6\left(\frac{Gm}{v}\right)^2\frac{v\,t\,N}{R^3}\ln\left(\frac{b_{max}}{b_{min}}\right) \quad ,$$

the required result. [Lectures] [4 marks]

The relaxation time T_{relax} is defined as the time for the changes in v^2 to reach v^2. Therefore, $\Delta v^2(T_{relax}) = v^2$. Substituting for v^2 into this,

$$6\left(\frac{Gm}{v}\right)^2\frac{v\,T_{relax}\,N}{R^3}\ln\left(\frac{b_{max}}{b_{min}}\right) = v^2 \quad .$$

Rearranging,

$$T_{relax} = \frac{1}{6N\ln\left(\frac{b_{max}}{b_{min}}\right)}\frac{(Rv)^3}{(Gm)^2} \quad ,$$

the required result. [Lectures] [4 marks]

[Total 20 marks for question]

3. (a) The distribution function f expresses the density of stars in the six-dimensional phase space defined by position and velocity in a galaxy. [Lectures] [2 marks]

(b) From the collisionless Boltzmann equation, f is constant. As the density of stars in space increases, the density in velocity space must decrease, to keep f constant. Therefore the velocity dispersions must increase. [Lectures] [3]

(c) Try $f = b\,(-E_m)^{7/2}$, where E_m is the energy per unit mass and b is a constant. The density becomes

$$\rho(r) \;=\; 4\pi\,\sqrt{2}\,\overline{m}\,b \int_{\Phi}^{0} \sqrt{E_m - \Phi}\;(-E_m)^{7/2}\,\mathrm{d}E_m \;. \qquad [2]$$

(Note that E_m and Φ are both negative, so $-E_m$ and $-\Phi$ are positive.) Use the substitution $E_m = \Phi\cos^2\theta$. Differentiating, $\mathrm{d}E_m = -2\Phi\sin\theta\cos\theta\,\mathrm{d}\theta$. The limits of the integral are $\theta = 0$ when $E_m = \Phi$, and $\theta = \pi/2$ when $E_m = 0$. Using this substitution, the density becomes

$$\begin{aligned}
\rho(r) &= 4\pi\,\sqrt{2}\,\overline{m}\,b \int_0^{\pi/2} \sqrt{\Phi\cos^2\theta - \Phi}\;(-\Phi\cos^2\theta)^{7/2}\;(-2\Phi\sin\theta\cos\theta\mathrm{d}\theta) \\
&= 4\pi\,\sqrt{2}\,\overline{m}\,b \int_0^{\pi/2} \sqrt{-\Phi}\;\sin\theta\,(-\Phi)^{7/2}\;\cos^7\theta\,(2)(-\Phi)\;\sin\theta\;\cos\theta\;\mathrm{d}\theta \\
&= 8\pi\,\sqrt{2}\,\overline{m}\,b\,(-\Phi)^5 \int_0^{\pi/2} \sin^2\theta\;\cos^8\theta\;\mathrm{d}\theta \;. \qquad [4]
\end{aligned}$$

Using the standard integral, we get,

$$\rho(r) \;=\; 8\pi\,\sqrt{2}\,\overline{m}\,b\,(-\Phi)^5 \left(\frac{7\pi}{512}\right) \;=\; \frac{7\,\sqrt{2}\,\pi^2}{64}\,\overline{m}\,b\,(-\Phi)^5 \;.$$

Substituting for the potential in the question, $\Phi(r) = -GM_{tot}/\sqrt{r^2 + a^2}$, we obtain,

$$\rho(r) \;=\; \frac{7\,\sqrt{2}\,\pi^2}{64}\,\overline{m}\,b\,\frac{G^5\,M_{tot}^5}{(r^2 + a^2)^{5/2}} \;.$$

This is the same as the expression for the density in the question if

$$b \;=\; \frac{24\sqrt{2}}{7\,\pi^3}\,\frac{a^2}{\overline{m}\,G^5\,M_{tot}^4} \;, \qquad \text{a constant} \;.$$

So $f(-E_m) = b(-E_m)^{7/2}$ is a solution to the equation relating density and the distribution function in the question if the constant b has this value.

[Seen in example problem] [3]

(d) The galaxy has a flattened potential: the orbit shows a 'rosette' pattern in the $x - y$ plane, but the rosette orbit is not confined to a plane. In such a potential, the orbit approximately shows a rosette pattern in a plane, but this plane precesses about the axis of the flattened potential. [Lectures] [4]

(e) Violent relaxation is the process by which the orbits of stars are altered, and randomised, by rapid changes in the gravitational potential. [Lectures] [2]

[Total 20 marks for question]

4. (a) The collisionless Boltzmann equation expresses the changes in the distribution function f as a function of time, position and velocity. The distribution function is difficult to determine for any galaxy observationally. In contrast, the Jeans equations use parameters such as number densities, mean velocities, mean square velocities and velocity dispersions that are much more easily determined from observations.

[Lectures] [2]

(b) The Jeans equation required has its summation over j. So begin by changing the summation in the collisionless Boltzmann equation given to be a summation over j (not over i as given). Take the first moment of the collisionless Boltzmann equation by multiplying by v_i and integrating over all velocities. Multiplying the C.B.E. throughout by v_i and using $v_j = \mathrm{d}x_j/\mathrm{d}t$ and $\mathrm{d}v_j/\mathrm{d}t = -\,\partial\Phi/\partial x_j$, we obtain,

$$v_i\,\frac{\partial f}{\partial t} + v_i\sum_{j=1}^{3} v_j\frac{\partial f}{\partial x_j} - v_i\sum_{j=1}^{3}\frac{\partial \Phi}{\partial x_j}\frac{\partial f}{\partial v_j} \;=\; 0\;.$$

(switching the index of the summation from i to j). [Lectures] [3 marks]

Integrating this over all velocities,

$$\int\left(v_i\,\frac{\partial f}{\partial t} + \sum_{j=1}^{3} v_i\,v_j\frac{\partial f}{\partial x_j} - \sum_{j=1}^{3} v_i\,\frac{\partial \Phi}{\partial x_j}\frac{\partial f}{\partial v_j}\right)\mathrm{d}^3\mathbf{v} \;=\; \int 0\,.\,\mathrm{d}^3\mathbf{v}\;.$$

$$\therefore\quad \int v_i\,\frac{\partial f}{\partial t}\,\mathrm{d}^3\mathbf{v} \;+\; \sum_{j=1}^{3}\int v_i\,v_j\,\frac{\partial f}{\partial x_j}\,\mathrm{d}^3\mathbf{v} \;-\; \sum_{j=1}^{3}\int v_i\,\frac{\partial \Phi}{\partial x_j}\,\frac{\partial f}{\partial v_j}\,\mathrm{d}^3\mathbf{v} \;=\; 0\;.$$

Noting that the integration over velocity is independent of the partial differentiation with respect to time t and to space coordinates x_j,

$$\frac{\partial}{\partial t}\int v_i\,f\,\mathrm{d}^3\mathbf{v} \;+\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\int v_i\,v_j\,f\,\mathrm{d}^3\mathbf{v} \;-\; \sum_{j=1}^{3}\int v_i\,\frac{\partial \Phi}{\partial x_j}\,\frac{\partial f}{\partial v_j}\,\mathrm{d}^3\mathbf{v} \;=\; 0\;.$$

[Lectures] [3 marks]

Substituting the terms given in the question, we obtain,

$$\frac{\partial(n\langle v_i\rangle)}{\partial t} \;+\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}(n\langle v_i v_j\rangle) \;-\; \sum_{j=1}^{3}\left(-\frac{\partial \Phi}{\partial x_i}\,\delta_{ij}\,n\right) \;=\; 0\;,$$

which gives the required Jeans equation. [Lectures] [2 marks]

(c) Because the galaxy has an isotropic velocity distribution and zero net rotation, $\langle v_r^2\rangle = \langle v_\theta^2\rangle = \langle v_\phi^2\rangle = \sigma^2$. The Jeans equation for spherical symmetry given in the information section,

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(n\,\langle v_r^2\rangle\Big) \;+\; \frac{n}{r}\Big[2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\Big] \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\;,$$

becomes $\dfrac{\mathrm{d}(n\sigma^2)}{\mathrm{d}r} + \dfrac{n}{r}\left(0\right) = -\,n\,\dfrac{\mathrm{d}\Phi}{\mathrm{d}r}$, and therefore, $\sigma^2\,\dfrac{\mathrm{d}n}{\mathrm{d}r} = -\,n\,\dfrac{\mathrm{d}\Phi}{\mathrm{d}r}$, because σ is constant across the galaxy. Integrating this from the centre to a radius r,

$$\sigma^2 \int_{n_0}^{n(r)} \frac{\mathrm{d}n}{n} = -\int_{\Phi(0)}^{\Phi(r)} \mathrm{d}\Phi \qquad \therefore\ \sigma^2\Big[\ln n\Big]_{n_0}^{n(r)} = -\Big[\Phi\Big]_{\Phi(0)}^{\Phi(r)}$$

$$\sigma^2\Big(\ln n(r) - \ln n_0\Big) = -\Big(\Phi(r) - \Phi(0)\Big) \quad \text{and so} \quad \sigma^2 \ln\left(\frac{n(r)}{n_0}\right) = -\,\Phi(r) + \Phi(0)\ .$$

Using $\Phi(r) = \dfrac{GM_{tot}}{(r+a)}$, $\quad \sigma^2 \ln\left(\dfrac{n(r)}{n_0}\right) = \dfrac{GM_{tot}}{r+a} - \dfrac{GM_{tot}}{a} = -\,\dfrac{GM_{tot}\,r}{a(r+a)}$

on simplifying. This then rearranges to give

$$n(r) = n_0\,\exp\left(-\,\frac{GM_{tot}\,r}{\sigma^2\,a(r+a)}\right)\ .$$

[New application of principles discussed in lectures] [6 marks]

(d) The kinetic energy of a star of mass m is $T_{star} = \frac{1}{2}mv^2$. The total kinetic energy of all stars is $T = \Sigma\frac{1}{2}mv^2 = \frac{1}{2}M_{tot}v^2$ on assuming that the galaxy contains only stars (so that M_{tot} is the sum of the masses of all stars). Using the virial theorem result $2\langle T\rangle + \langle U\rangle = 0$ and the expression for the total internal potential energy in the question,

$$2\left(\tfrac{1}{2}M_{tot}v^2\right) - \frac{GM_{tot}^2}{6a} = 0\ .$$

Rearranging, $\quad M_{tot} = \dfrac{6av^2}{G}\ .$

[Unseen application of principles from lectures] [4 marks]

[Total 20 marks for question]

5. (a) The rotation curve has maxima at $R \simeq 2.9R_S$ and at $R \gtrsim 12R_S$. The maximum at $R \simeq 2.9R_S$ is due to the stellar exponential disc: pure exponential discs show maxima in the rotation curve at $R \simeq 2.5 - 3.0$ times their own scale lengths. The maximum at $R \gtrsim 12R_S$ is due to a dark matter halo. Therefore the disc and dark matter halo show dynamical signatures in the rotation curve. A bulge does not.

[Lectures] [6 marks]

The observations extend to $R \simeq 12$ disc scale lengths. Optical spectroscopy could not measure rotation velocities at these faint surface brightness levels. The rotation curve must therefore have been made using radio 21 cm observations. [Lectures] [2]

(b) For a thin spherical shell we have, $\mathrm{d}M(r) = 4\pi r^2\rho(r)\mathrm{d}r$. Integrating from the centre ($r = 0$) to a radial distance r, we obtain,

$$M(r) = 4\pi\rho_0 \int_0^r \frac{r'^2}{1 + r'^2/a^2}\,\mathrm{d}r' = 4\pi\rho_0 a^2 \int_0^r \frac{r'^2}{r'^2 + a^2}\,\mathrm{d}r'\ .$$

Using the standard integral given in the Information Section,

$$M(r) = 4\pi\rho_0 a^2\left[r' - a\tan\left(\frac{r'}{a}\right)\right]_0^r = 4\pi\rho_0\,a^2\left(r - a\tan^{-1}(r/a) - \tan^{-1}0 + 0\right)$$

$$\therefore \quad M(r) \;=\; 4\pi\rho_0\, a^2 \left(r \;-\; a\tan^{-1}(r/a) \right) .$$

[Seen in example problem] [6]

At a radial distance r from the centre, using the spherical symmetry,

$$\frac{GM(r)}{r^2} \;=\; \frac{v_{circ}^2}{r} \qquad \therefore \quad v_{circ} \;=\; \sqrt{4\pi G\rho_0\, a^2 \left(1 \;-\; \frac{a}{r}\tan^{-1}(r/a)\right)}$$

When $r \gg a$, $\tan^{-1}(r/a) \simeq \pi/2$, so $\frac{a}{r}\tan^{-1}(r/a) \ll 1$.

$$\therefore \quad v_{circ} \;=\; \sqrt{4\pi G\rho_0\, a^2} \;=\; \text{constant} .$$

[Application of principles discussed in lectures] [2]

This model implies a flat rotation curve far from the centre ($r \gg a$). The Galaxy is indeed observed to have a flat rotation curve out to large radii. [Lectures] [2]

The total mass implied is $M_{tot} = \lim_{r\to\infty} M(r) \to \infty$. So the total mass is infinite. The real density profile must fall below this $\rho(r) = \rho_0/(1 + r^2/a^2)$ at large radii.

[Lectures] [2]

[Total 20 marks for question]

6. (a) The hydrogen lines are produced by electron transitions in hydrogen atoms down to the $n = 2$ energy level from a higher level: they are hydrogen Balmer lines. The atoms are formed by the recombination of free protons and free electrons in the H II region, and these atoms on recombination have their electrons in excited energy states. The electrons cascade down through the energy levels until they emit the Balmer photons as they reach the $n = 2$ state. The underlying energy source is ultraviolet radiation from hot stars that produces the ionisation.

The [O II] and [O III] lines are produced by oxygen ions that have been collisionally excited by interactions with other ions. [Lectures] [4 marks]

(b) Maser emission occurs when radiation from a suitable source produces a population inversion between certain electron energy levels in nearby gas: some higher energy levels have more electrons than lower energy levels. The overpopulated excited state then decays by stimulated emission: it produces maser emission. [Lectures] [2]

(c) Gas is collisional in an astronomical context. [Lectures] [2]

(d) The observed colour index of the star is $B - V = 18.45 - 17.15 = 1.30$ mag. The colour excess is therefore $E_{B-V} = (B - V) - (B - V)_0 = 1.30 - 0.80 = 0.50$ mag. The extinction in the V-band will be given by $A_V \simeq 3.1E_{B-V}$. Therefore $A_V \simeq 3.1 \times 0.50 = 1.55$ mag. The distance of the star from the Earth, D, is related to the apparent magnitude V, the absolute V-band magnitude M_V and the extinction A_V by $V - M_V = 5\log_{10}(D/\text{pc}) - 5 + A_V$ (given in the question paper's information section). Rearranging,

$$5\log_{10}(D/\text{pc}) \;=\; V - M_V + 5 - A_V \;=\; 17.15 - 5.60 + 5 - 1.55 = 15.00$$

This gives, $\log_{10}(D/\text{pc}) \;=\; 3.00$ and therefore, the distance of the star from the Earth is $D = 1000$ pc. [New application of principles seen in coursework] [4]

If the longitude of the star is 180° and it lies in the Galactic plane, its distance from the Galactic Centre is $R = R_0 + 1000\,\text{pc} = 8.0\,\text{kpc} + 1000\,\text{pc} = 9.0\,\text{kpc}$.

[Unseen] [1]

(e) The surface mass density of stars can be obtained by integrating the density over height z:

$$\Sigma_s \;=\; \int_{-\infty}^{\infty} \rho_s(z)\,\mathrm{d}z \;=\; \int_{-\infty}^{\infty} \rho_{so}\,\mathrm{e}^{-|z|/h_s}\,\mathrm{d}z \;=\; 2\int_0^{\infty} \rho_{so}\,\mathrm{e}^{-z/h_s}\,\mathrm{d}z \quad \text{(from symmetry)}$$

$$= \; 2\,\rho_{so}\int_0^{\infty} \mathrm{e}^{-z/h_s}\,\mathrm{d}z \;=\; 2\,\rho_{so}\,\left[-h_s\,\mathrm{e}^{-z/h_s}\right]_{z=0}^{\infty} \;=\; 2\,\rho_{so}\,h_s \;.$$

Similarly, for the gas, $\Sigma_g \;=\; 2\,\rho_{go}\,h_g$. Therefore,

$$\frac{\Sigma_s}{\Sigma_g} \;=\; \frac{2\,\rho_{so}\,h_s}{2\,\rho_{go}\,h_g} \;=\; \frac{\rho_{so}}{\rho_{go}}\,\frac{h_s}{h_g} \;=\; 6 \times \frac{250}{150} \;=\; 10 \;.$$

So $\Sigma_s = 10\,\Sigma_g$ at the Sun's distance from the Galactic Centre. [Coursework] [4]
Dust density ρ_d closely follows that of gas and observations show that $\rho_d/\rho_g \simeq 0.1$. Therefore we expect $\Sigma_s \simeq 100\,\Sigma_d$ at the Sun's distance from the Galactic Centre.

[Coursework] [1]

In practice, the density of gas is very patchy. While the $\rho_g(z) = \rho_{go}\mathrm{e}^{-|z|/h_g}$ law may represent the broad trend of gas density with distance from the Galactic plane, it is a poor representation of the detailed behaviour of the gas density. [Unseen] [2]

[Total 20 marks for question]

7. (a) Assumptions behind the Simple Model: (i) the volume of space is a 'closed box' (no gas enters or leaves the volume); (ii) the volume initially contains only unenriched gas (gas initially of zero metallicity and no stars); (iii) the gas is well mixed (the same chemical composition throughout); (iv) instantaneous recycling occurs; (v) the fraction of newly created heavy elements ejected into the gas when gas forms stars is constant (constant yield). [Lectures] [1 mark for each, up to a max. of 4]

(b) The metallicity $Z \equiv M_{\text{metals}}/M_{\text{gas}}$, i.e. $Z = \text{fn}(M_{\text{metals}}, M_{\text{gas}})$. Differentiating this with respect to time t,

$$\frac{\mathrm{d}Z}{\mathrm{d}t} \;=\; \frac{\partial Z}{\partial M_{\text{metals}}}\,\frac{\mathrm{d}M_{\text{metals}}}{\mathrm{d}t} \;+\; \frac{\partial Z}{\partial M_{\text{gas}}}\,\frac{\mathrm{d}M_{\text{gas}}}{\mathrm{d}t} \;.$$

Calculating the partial differentials,

$$\frac{\partial Z}{\partial M_{\text{metals}}} \;=\; \frac{1}{M_{\text{gas}}} \quad \text{and} \quad \frac{\partial Z}{\partial M_{\text{gas}}} \;=\; -\,\frac{M_{\text{metals}}}{M_{\text{gas}}^2} \;,$$

which gives,

$$\frac{\mathrm{d}Z}{\mathrm{d}t} \;=\; \frac{1}{M_{\text{gas}}}\,\frac{\mathrm{d}M_{\text{metals}}}{\mathrm{d}t} \;-\; \frac{M_{\text{metals}}}{M_{\text{gas}}^2}\,\frac{\mathrm{d}M_{\text{gas}}}{\mathrm{d}t} \;.$$

[Lectures] [4]

For a small time interval δt, we have,

$$\delta Z \;=\; \frac{\delta M_{\text{metals}}}{M_{\text{gas}}} \;-\; \frac{M_{\text{metals}}}{M_{\text{gas}}^2}\,\delta M_{\text{gas}} \;=\; \frac{\delta M_{\text{metals}}}{M_{\text{gas}}} \;-\; Z\,\frac{\delta M_{\text{gas}}}{M_{\text{gas}}}$$

on subsituting $M_{\rm metals}/M_{\rm gas} \equiv Z$, giving the required result. [Lectures] [4 marks]
The volume in the Simple Model is a closed box, so $M_{\rm stars} + M_{\rm gas} = M_{\rm total}$, a constant. Therefore, $\delta M_{\rm stars} + \delta M_{\rm gas} = 0$. Subsituting $\delta M_{\rm stars} = -\,\delta M_{\rm gas}$ into $\delta M_{\rm metals} = -\,Z\,\delta M_{\rm stars} + p\,\delta M_{\rm stars}$ and dividing by $M_{\rm gas}$,

$$\frac{\delta M_{\rm metals}}{M_{\rm gas}} = Z\,\frac{\delta M_{\rm gas}}{M_{\rm gas}} - p\,\frac{\delta M_{\rm gas}}{M_{\rm gas}} .$$

Substituting for $\delta M_{\rm metals}/M_{\rm gas}$ into the expression for δZ derived above,

$$\delta Z = Z\,\frac{\delta M_{\rm gas}}{M_{\rm gas}} - p\,\frac{\delta M_{\rm gas}}{M_{\rm gas}} - Z\,\frac{\delta M_{\rm gas}}{M_{\rm gas}} = -\,p\,\frac{\delta M_{\rm gas}}{M_{\rm gas}} ,$$

the required result. [Lectures] [3 marks]
Converting this into a differential equation and integrating from time 0 to t,

$$\int_0^{Z(t)} {\rm d}Z' = -\int_{M_{\rm gas}(0)}^{M_{\rm gas}(t)} p\,\frac{{\rm d}M'_{\rm gas}}{M'_{\rm gas}}$$

$$\therefore\quad Z(t) = -\,p\,\left[\,\ln M'_{\rm gas}\right]_{M_{\rm gas}(0)}^{M_{\rm gas}(t)} = -\,p\,\ln\left(\frac{M_{\rm gas}(t)}{M_{\rm gas}(0)}\right) = -\,p\,\ln\mu ,$$

the required result, because $M_{\rm gas}(0) = M_{\rm total}$, a constant, and so $\mu \equiv M_{\rm gas}(t)/M_{\rm total} = M_{\rm gas}(t)/M_{\rm gas}(0)$. [Lectures] [4 marks]
Observations of the gas in galaxies can provide the Z of the gas today at some location, and the local gas fraction μ. Irregular galaxies fit this relation moderately well, given a suitable choice of the yield p. In spiral galaxies, the gas fraction in the disc increases with distance from the centre and Z is indeed observed to decrease, though perhaps more steeply than this model predicts. [Lectures] [3 marks]

(c) [Fe/H] $= -2.0$ indicates a very metal-poor star, typical of the stellar halo of the Galaxy. It will be very old ($\simeq 13$ Gyr). [Lectures] [2 marks]

[Total 20 marks for question]

8. (a) The solid angle subtended by an Einstein ring is $\pi\,\theta_E^2$. Consider a field subtending a solid angle Ω. The fraction of this field covered by Einstein rings of lensing objects at distances between D_L and $D_L + {\rm d}D_L$ is ${\rm d}\tau = \pi\,\theta_E^2\,{\rm d}N\,/\,\Omega$ where ${\rm d}N$ is the number of lenses in this thin volume, and τ is the optical depth. If n is the number density of lenses, ${\rm d}N = n \times$ (surface area of shell) $\times$ thickness $= n(D_L^2\Omega)\,{\rm d}D_L$. The mass density is $\rho = nM_L$. Therefore, $n = \rho/M_L$, and so,

$${\rm d}\tau = \frac{\pi\,\theta_E^2}{\Omega}\,\frac{\rho}{M_L}\,(\Omega D_L^2)\,{\rm d}D_L = \frac{\pi\,\theta_E^2\,\rho\,D_L^2\,{\rm d}D_L}{M_L} .$$

Substituting for the angular Einstein radius and integrating over distance,

$$\tau = \int_0^{D_S} \pi\left(\frac{4GM_L}{c^2}\,\frac{D_{LS}}{D_L D_S}\right)\frac{\rho D_L^2}{M_L}\,{\rm d}D_L = \frac{4\pi G}{c^2 D_S}\int_0^{D_S} D_L\,D_{LS}\,\rho(D_L)\,{\rm d}D_L ,$$

the required result. [Lectures] [6 marks]

(b) Use the equation derived in part (a). Considering the geometry, $D_S = R_0$, $D_{LS} = R_0 - D_L$ and also $D_{LS} = r$. Therefore, $r = R_0 - D_L$. The optical depth is then

$$\begin{aligned}\tau &= \frac{4\pi G}{c^2 R_0} \int_0^{R_0} D_L\,(R_0 - D_L)\, \frac{\rho_0\, a^2}{(R_0 - D_L)^2 + a^2}\, \mathrm{d}D_L \\ &= \frac{4\pi G \rho_0 a^2}{c^2 R_0} \int_0^{R_0} \frac{D_L\,(R_0 - D_L)}{(R_0 - D_L)^2 + a^2}\, \mathrm{d}D_L \\ &= \frac{4\pi G \rho_0 a^2}{c^2 R_0} \left[-D_L \;-\; a \tan^{-1}\left(\frac{R_0 - D_L}{a}\right) \;-\; \frac{1}{2} R_0 \ln\left(a^2 + (R_0 - D_L)^2\right)\right]_0^{R_0}\end{aligned}$$

using the standard integral given in the question. [Coursework] [6]

This gives,

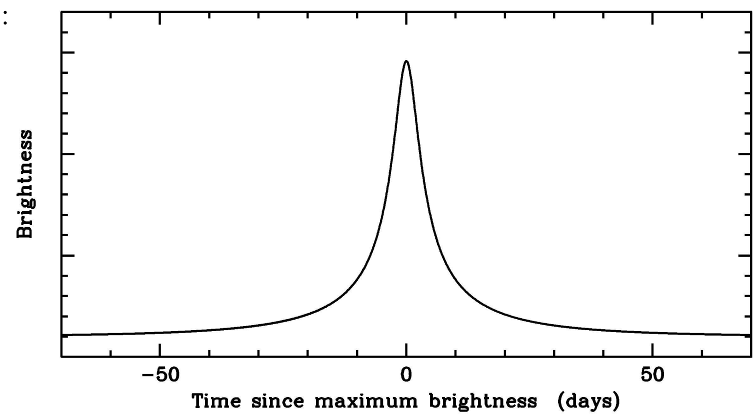
$$\begin{aligned}\tau &= \frac{4\pi G \rho_0 a^2}{c^2 R_0} \Bigg(-R_0 \;-\; a \tan^{-1}(0) \;-\; \frac{1}{2} R_0 \ln\left(a^2 + (R_0 - R_0)^2\right) \;+\; 0 \\ &\qquad\qquad + a \tan^{-1}\left(\frac{R_0}{a}\right) \;+\; \frac{1}{2} R_0 \ln\left(a^2 + R_0^2\right) \Bigg) \\ &= \frac{2\pi G \rho_0 a^2}{c^2} \left(\ln\left(1 + \frac{R_0^2}{a^2}\right) \;+\; \frac{2a}{R_0} \tan^{-1} \frac{R_0}{a} \;-\; 2 \right) ,\end{aligned}$$

the required result. [2]

Substituting $\rho_0 = 2.0 \times 10^{-20}\,\mathrm{kg\,m^{-3}}$, $a = 2.0 \times 10^3\,\mathrm{pc} = 2.0 \times 10^3 \times 3.086 \times 10^{16}\,\mathrm{m}$ and $R_0/a = 4.0$ gives $\tau = 5.3 \times 10^{-7}$. [2]

To be able to detect microlensing events in practice would require monitoring the brightnesses of very large numbers of stars, say $> 10^7$ stars. [1]

(c) A typical microlensing event:



[Lectures] [3]

[Total 20 marks for question]