ASTM002 / MAS430 The Galaxy

Summer 2005 Examination Paper Model Answers

1. (a) Elliptical galaxies show only absorption lines, characteristic of old stars. Irregular galaxies show very strong emission lines from H II regions superimposed on a stellar continuum characteristic of moderately young stars. [Lectures] [2 marks]

(b) Hydrogen [1 mark]; oxygen and nitrogen [1 mark for any one of these, up to a maximum of 2]. [Lectures] [2]

Forbidden lines are produced by electron energy transitions with low transition probabilities, and therefore from states with long lifetimes. Under laboratory conditions these excited states are collisionally de-excited before the transitions can occur. In contrast, collisions are so infrequent at the low densities in the interstellar medium that radiative transitions can occur. [Lectures] [2]

One Balmer photon is produced by the interstellar gas from each Lyman continuum photon from the star [because a large majority of the H atoms in the gas are in the ground state]. Summing the Balmer emission, the Balmer luminosity of the H II region is 2.6×10^{48} photons s^{-1}. The star must therefore produce 2.6×10^{48} photons s^{-1} shortward of the Lyman limit. The wavelength of 912 Å is that of the Lyman limit: photons shortwards of this ionise hydrogen atoms from the ground state.

[Application of principles from lectures] [3]

(c) CO is most easily observed at millimetre wavelengths (radio) [emitting at 1.3 and 2.6 mm]. This emission is produced by rotational transitions of the molecules.

[Lectures] [2]

H_2 produces minimal emission and no radio lines. CO does emit readily at millimetre wavelengths and its distribution can be mapped directly. It is therefore used to trace the distribution of cold molecular gas. [Lectures] [2]

(d) The observed colour index is $(B-V) = 16.20 - 15.30 = 0.90$ mag. Therefore the $(B-V)$ colour excess is $E_{B-V} \equiv (B-V) - (B-V)_0 = 0.90 - 0.60 = 0.3$ mag. The extinction A_V in the V band is approximately $A_V \simeq 3.3 E_{B-V}$, giving $A_V = 1.0$ mag.

[Unseen, but principles given in lectures and in an example problem] [3]

(e) The optical depth when travelling a distance $\mathrm{d}l$ along the line of sight at galactic latitude b is $\mathrm{d}\tau = \kappa_\lambda\,\rho(z)\,\mathrm{d}l = \kappa_\lambda\,\rho(z)\,\mathrm{d}z/\sin|b|$ where z is the distance north of the Galactic plane. Integrating along the line of sight,

$$\int_0^{\tau_\lambda} \mathrm{d}\tau'_\lambda = \int_0^\infty \frac{\kappa_\lambda\,\rho(z)\,\mathrm{d}z}{\sin|b|} = \int_0^\infty \frac{\kappa_\lambda\,\rho_0\,\mathrm{e}^{-|z|/h}}{\sin|b|}\,\mathrm{d}z = \frac{\kappa_\lambda\,\rho_0}{\sin|b|}\int_0^\infty \mathrm{e}^{-|z|/h}\,\mathrm{d}z$$

$$\therefore \quad \tau_\lambda = \frac{\kappa_\lambda\,\rho_0\,h}{\sin|b|} = \kappa_\lambda\,\rho_0\,h\,\mathrm{cosec}|b|$$

Since the extinction is $A_\lambda = 1.086\tau_\lambda$, we get $A_\lambda = 1.086\,\kappa_\lambda\,\rho_0\,h\,\mathrm{cosec}|b|$.

[Unseen] [3]

(f) The total mass in stars and gas is constant in the closed volume, i.e. $M_{stars} + M_{gas} = M_{total}$, a constant. So the change in the mass of stars is $\delta M_{stars} = -\delta M_{gas}$. Substituting for $\delta M_{metals}/M_{gas}$ from the first equation into the second, and for

$\delta M_{stars} = -\delta M_{gas}$, we obtain $\delta Z = -p\,\delta M_{gas}/M_{gas}$, from which we derive

$$\int_0^{Z(t)} \mathrm{d}Z' \;=\; -\,p \int_{M_{total}}^{M_{gas}(t)} \frac{\mathrm{d}M'_{gas}}{M'_{gas}}\,,$$

because initially the volume contained only zero metallicity gas (so initially $Z = 0$ and $M_{gas} = M_{total}$). This gives $Z = -\,p\ln(M_{gas}(t)/M_{total}) = -\,p\ln\mu$ if $\mu \equiv M_{gas}(t)/M_{total}$. [Lectures] [5]
Irregular galaxies fit this relation moderately well, given a suitable choice of the yield p. [Lectures] [1]

2. (a) A system of stars in a galaxy that is pressure supported has star orbits that are predominantly randomly oriented. A system of stars that is rotationally supported predominantly has stars that orbit in similar directions, and therefore has a net angular momentum. [Principles given in lectures] [2 marks]

(b) For a thin spherical shell we have, $\mathrm{d}M(r) = 4\pi r^2\rho(r)\mathrm{d}r$. Integrating from the centre ($r = 0$) to a radial distance r, we obtain,

$$M(r) \;=\; 4\pi\rho_0 \int_0^r \frac{r'^2}{1 + r'^2/a^2}\,\mathrm{d}r' \;=\; 4\pi\rho_0 a^2 \int_0^{\tan^{-1}(r/a)} \frac{\tan^2\theta}{1+\tan^2\theta}\, a\,\sec^2\theta\,\mathrm{d}\theta$$

on making the $r' = a\tan\theta$ substitution. Using the identity $1 + \tan^2\theta \equiv \sec^2\theta$,

$$M(r) \;=\; 4\pi\rho_0 a^3 \int_0^{\tan^{-1}(r/a)} \tan^2\theta\,\mathrm{d}\theta \;=\; 4\pi\rho_0\,a^3\left[\tan(\theta) \;-\; \theta\right]_0^{\tan^{-1}(r/a)}$$

$$= \; 4\pi\rho_0\,a^3\,\left(r/a \;-\; \tan^{-1}(r/a) \;-\; \tan 0 \;+\; 0\right)$$

$$\therefore\;\; M(r) \;=\; 4\pi\rho_0\,a^2\,\left(r \;-\; a\tan^{-1}(r/a)\right)\;.$$

[Seen in example problem] [6]

(c) At a radial distance r from the centre, using the spherical symmetry,

$$\frac{GM(r)}{r^2} \;=\; \frac{v_{circ}^2}{r} \qquad\qquad \therefore\;\; v_{circ} \;=\; \sqrt{4\pi G\rho_0\,a^2\,\left(1 \;-\; \frac{a}{r}\tan^{-1}(r/a)\right)}$$

When $r \gg a$, $\tan^{-1}(r/a) \simeq \pi/2$, so $\frac{a}{r}\tan^{-1}(r/a) \ll 1$.

$$\therefore\;\; v_{circ} \;=\; \sqrt{4\pi G\rho_0\,a^2} \;=\; \text{constant}\,.$$

[Application of principles discussed in lectures] [3]
This model implies a flat rotation curve far from the centre ($r \gg a$). The Galaxy is indeed observed to have a flat rotation curve out to large radii. [Lectures] [2]

(d) We have $M(r) \to \infty$ as $r \to \infty$. This is physically unrealistic. So the real density profile must fall below the model profile at large radii (through a steepening of the density law or a truncation of the mass distribution).

[Application of principles discussed in lectures] [3]

(e) For this potential because of the spherical symmetry (i.e. because $\partial\Phi/\partial\theta = 0$ and $\partial\Phi/\partial\phi = 0$),

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = -\frac{GM_{tot}}{a}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\frac{r+a}{r}\right)\right)$$

$$= -\frac{GM_{tot}}{a}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(-r^2\frac{a}{r(r+a)}\right) = \frac{GM_{tot}\,a}{r^2(r+a)^2} \ .$$

Using $\rho = \nabla^2\Phi/4\pi G$,

$$\rho(r) = \frac{M_{tot}}{4\pi}\frac{a}{r^2(r+a)^2} \ .$$ [4]

As $r \to \infty$, $\Phi(r) \to 0$. This is physically realistic: the potential is zero at large radius.

As $r \to 0$, $\Phi(r) \to -\infty$. This is physically unrealistic: in reality the potential will be be finite (and negative) at the centre.

[Application of principles from lectures to a new example] [2]

(f) The radial model assumes that the velocities of the galaxies in space are the observed radial velocities. The model having transverse velocity components assigns space velocities greater than the observed radial velocities, and therefore velocities greater than the radial model. The deduced mass will be larger in the model having transverse velocity components. [Unseen] [3]

3. (a) The distribution function f expresses the density of stars in the six-dimensional phase space defined by position and velocity in a galaxy. [Lectures] [2 marks]

(b) In the axisymmetric potential, the total energy of a star and the component of the angular momentum parallel to the axis of symmetry are constant: they are therefore integrals of motion. The total angular momentum is not constant: it is not an integral of motion. [Lectures] [3 marks]

(c) Try $f = b\,(-E_m)^{7/2}$, where E_m is the energy per unit mass and b is a constant. The density becomes

$$\rho(r) = 4\pi\sqrt{2}\,\overline{m}\,b\int_{\Phi}^{0}\sqrt{E_m - \Phi}\,(-E_m)^{7/2}\,\mathrm{d}E_m \ .$$

Note that E_m and Φ are both negative, so $-E_m$ and $-\Phi$ are positive. Use the substitution $E_m = \Phi\cos^2\theta$. Differentiating, $\mathrm{d}E_m = -2\Phi\sin\theta\cos\theta\,\mathrm{d}\theta$. The limits of the integral are $\theta = 0$ when $E_m = \Phi$, and $\theta = \pi/2$ when $E_m = 0$. Using this substitution, the density becomes

$$\begin{aligned}
\rho(r) &= 4\pi\,\sqrt{2}\;\overline{m}\,b \int_0^{\pi/2} \sqrt{\Phi\cos^2\theta - \Phi}\;\left(-\Phi\cos^2\theta\right)^{7/2}\;\left(-2\Phi\sin\theta\cos\theta \mathrm{d}\theta\right) \\
&= 4\pi\,\sqrt{2}\;\overline{m}\,b \int_0^{\pi/2} \sqrt{-\Phi}\;\sin\theta\,(-\Phi)^{7/2}\;\cos^7\theta\;(2)(-\Phi)\;\sin\theta\;\cos\theta\;\mathrm{d}\theta \\
&= 8\pi\,\sqrt{2}\;\overline{m}\,b\,(-\Phi)^5 \int_0^{\pi/2} \sin^2\theta\;\cos^8\theta\;\mathrm{d}\theta\;.
\end{aligned}$$

Using the standard integral, we get,

$$\rho(r) \;=\; 8\pi\,\sqrt{2}\;\overline{m}\,b\,(-\Phi)^5 \left(\frac{7\pi}{512}\right) \;=\; \frac{7\,\sqrt{2}\,\pi^2}{64}\;\overline{m}\,b\,(-\Phi)^5$$

Substituting for the potential in the question, we obtain,

$$\rho(r) \;=\; \frac{7\,\sqrt{2}\,\pi^2}{64}\;\overline{m}\,b\;\frac{G^5\,M_{tot}^5}{(r^2+a^2)^{5/2}}\;.$$

This is the same as the expression for the density in the question if

$$b \;=\; \frac{24\sqrt{2}}{7\,\pi^3}\;\frac{a^2}{\overline{m}\,G^5\,M_{tot}^4}\;.$$

So $f(-E_m) = b(-E_m)^{7/2}$ is a solution to the equation relating density and the distribution function in the question if the constant b has this value.

[Seen in example problem] [7]

(d) The collisionless Boltzmann equation expresses the changes in the distribution f as a function of time, position and velocity. The distribution function is difficult to determine for any galaxy observationally. In contrast, the Jeans equations use parameters such as number densities, mean velocities, mean square velocities and velocity dispersions that are much more easily determined from observations.

[Lectures] [2]

(e) By substituting for the components of acceleration from $\mathrm{d}\mathbf{v}/\mathrm{d}t = -\boldsymbol{\nabla}\Phi$, we can express the collisionless Boltzmann equation as

$$\frac{\partial f}{\partial t} \;+\; \sum_{i=1}^{3} v_i\,\frac{\partial f}{\partial x_i} \;-\; \sum_{i=1}^{3} \frac{\partial\Phi}{\partial x_i}\frac{\partial f}{\partial v_i} \;=\; 0\,,$$

To derive the first of the Jeans Equations, integrate this equation over all velocities at a point

$$\int\left(\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i\,\frac{\partial f}{\partial x_i} - \sum_{i=1}^{3} \frac{\partial\Phi}{\partial x_i}\frac{\partial f}{\partial v_i}\right)\mathrm{d}^3\mathbf{v} \;=\; \int 0\,.\,\mathrm{d}^3\mathbf{v}\;.$$

$$\therefore\quad \int \frac{\partial f}{\partial t}\,\mathrm{d}^3\mathbf{v} \;+\; \sum_{i=1}^{3}\int v_i\frac{\partial f}{\partial x_i}\,\mathrm{d}^3\mathbf{v} \;-\; \sum_{i=1}^{3}\int \frac{\partial\Phi}{\partial x_i}\frac{\partial f}{\partial v_i}\,\mathrm{d}^3\mathbf{v} \;=\; 0\;.$$

Because the integration is performed over all velocities for a given position and time,

$$\int \frac{\partial f}{\partial t}\, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial}{\partial t} \int f\, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial n}{\partial t} \,,$$

using $n = \int f\, \mathrm{d}^3\mathbf{v}$. Similarly,

$$\int v_i \,\frac{\partial f}{\partial x_i}\, \mathrm{d}^3\mathbf{v} \;=\; \int \frac{\partial (v_i f)}{\partial x_i}\, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial}{\partial x_i} \int v_i\, f\, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial\, (n\,\langle v_i\rangle\,)}{\partial x_i} \,,$$

on substituting $n\langle v_i\rangle = \int v_i\, f\, \mathrm{d}^3\mathbf{v}$. We also have

$$\int \frac{\partial \Phi}{\partial x_i}\,\frac{\partial f}{\partial v_i}\, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i}\, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial \Phi}{\partial x_i}\,(0) \;=\; 0 \,,$$

because $f \longrightarrow 0$ as $|v_i| \longrightarrow \infty$. Substituting for these terms,

$$\frac{\partial n}{\partial t} \;+\; \sum_{i=1}^{3} \frac{\partial n\langle v_i\rangle}{\partial x_i} \;=\; 0 \,,$$

the required result. [Lectures] [5]

(f) Because there is no net rotation and the velocity dispersion σ is constant and isotropic, $\langle v_r^2\rangle = \langle v_\theta^2\rangle = \langle v_\phi^2\rangle = \sigma^2$. The Jeans equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(n\,\sigma^2\Big) \;+\; \frac{n}{r}\,(\,0\,) \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} \qquad\qquad \therefore\quad \sigma^2\,\frac{\mathrm{d}n}{\mathrm{d}r} \;=\; -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Integrating,

$$\sigma^2 \int_{n_0}^{n(r)} \frac{\mathrm{d}n}{n} \;=\; -\int_{\Phi(0)}^{\Phi(r)} \mathrm{d}\Phi' \,.$$

$$\therefore\quad \sigma^2\,\Big[\,\ln n\,\Big]_{n_0}^{n(r)} \;=\; -\,\Phi(r) \;+\; \Phi(0)$$

$$\therefore\quad \sigma^2\,\Big(\,\ln n(r) \;-\; \ln n_0\Big) \;=\; \frac{GM_{tot}}{(r^2+a^2)^{1/2}} \;-\; \frac{GM_{tot}}{a}$$

$$\sigma^2\,\ln\left(\frac{n(r)}{n_0}\right) \;=\; \frac{GM_{tot}}{a}\left(\frac{1}{\sqrt{1+r^2/a^2}} \;-\; 1\right) \,,$$

which gives the required result.

[This is a more direct approach than differentiating the expression for Φ to substitute for $\mathrm{d}\Phi/\mathrm{d}r$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(n\,\sigma^2\Big) \;+\; \frac{n}{r}\,(\,0\,) \;=\; -\,n\,\frac{GM_{tot}\,r}{(r^2+a^2)^{3/2}} \,, \qquad\qquad \therefore\quad \sigma^2\,\frac{\mathrm{d}n}{\mathrm{d}r} \;=\; -\,n\,\frac{GM_{tot}\,r}{(r^2+a^2)^{3/2}} \,,$$

Integrating,

$$\sigma^2 \int_{n_0}^{n(r)} \frac{\mathrm{d}n}{n} \;=\; -\,GM_{tot}\int_0^r \frac{r'}{(r'^2+a^2)^{3/2}}\mathrm{d}r' \,.$$

$$\therefore \quad \sigma^2 \Big[\ln n \Big]_{n_0}^{n(r)} = -\,GM_{tot} \left[-\frac{1}{(r'^2 + a^2)^{1/2}} \right]_0^r .$$

$$\therefore \quad \sigma^2 \Big(\ln n(r) - \ln n_0 \Big) = GM_{tot} \left(\frac{1}{(r^2 + a^2)^{1/2}} - \frac{1}{a} \right)$$

which gives the required result.]

[Applying principles from lectures to a new potential] [6]

4. (a) The iron-to-hydrogen ratio by number, $N(\mathrm{Fe})/N(\mathrm{H})$, in the star is 1/100-th that in the Sun, i.e. a factor of $10^{[\mathrm{Fe/H}]}$. This comes from the definition of the [Fe/H] parameter: $[\mathrm{Fe/H}] \equiv \log(N(\mathrm{Fe})/N(\mathrm{H})) - \log(N(\mathrm{Fe})/N(\mathrm{H}))_\odot$, where $(N(\mathrm{Fe})/N(\mathrm{H}))_\odot$ is the iron-to-hydrogen ratio in the Sun. [Applying principles from lectures.] [1]

$[\mathrm{Fe/H}] = -1.54$ indicates that the star is very metal poor. It has a metallicity that is typical of the Galactic stellar halo. Stars in the stellar halo are very old. [Lectures] [2]

Type II supernovae explode $\sim 10^7$ yr after their progenitor stars were formed and eject large quantities of CNO elements relative to iron into the interstellar medium. Type Ia supernovae occur $\sim 10^8 - 10^9$ yr after their progenitor stars form and eject much larger quantities of iron relative to oxygen into the ISM. Therefore stars that formed within a short time of the start of a burst of star formation will be metal-poor but have high [O/Fe] compared to stars that form $> 10^8$ yr later. [Lectures] [3]

(b) Asymmetric drift is the lag in the azimuthal component of the mean velocity of a subpopulation of stars relative to the local standard of rest. [Lectures] [2]

Among the stars of the Galactic disc, there is a gradual increase in asymmetric drift with age. The velocity dispersion of disc stars increases with age. Encounters with giant molecular clouds will have perturbed the motions of old stars, increasing their velocity dispersions. [Lectures] [3]

(c) For a microlensing event to be detectable, the amplification of the solid angle of the background star by the lensing object must be large enough to make a measurable change in the flux from the star at the Earth.

Assuming the background star is at a distance $D_S = 50$ kpc $\simeq 50\,000 \times 3.1 \times 10^{16}$ m $\simeq 15 \times 10^{20}$ m and its radius is $R_* \simeq R_\odot \simeq 7 \times 10^8$ m, the angular radius of the star as observed from the Earth will be $\theta_* = R_*/D_S$ rad $\sim 5 \times 10^{-13}$ rad.

The brown dwarf has a mass $M \sim 0.05 M_\odot \sim 0.05 \times 2 \times 10^{30}$ kg $\sim 10^{29}$ kg. Assuming that it lies at a distance $D_L \simeq 20$ kpc $\simeq 20\,000 \times 3 \times 10^{16}$ m $\simeq 6 \times 10^{20}$ m, the lens–source distance will be $D_{LS} = D_S - D_L \simeq 9 \times 10^{20}$ m. Its Einstein angular radius from the Earth will be

$$\theta_E \sim \sqrt{\frac{4 \times 6.7 \times 10^{-11} \times 10^{29}}{(3 \times 10^8)^2} \, \frac{9 \times 10^{20}}{6 \times 10^{20} \times 15 \times 10^{20}}} \ \mathrm{rad} \sim 10^{-10}\ \mathrm{rad}\,.$$

This is significantly larger than the angular size of the star. Given a suitable alignment between the lensing object and the background star, there will be a measurable increase in the observed flux of light from the star.

The elementary particle has a mass $M \sim 10^{-24}$ kg. If it lies at the same distance, $D_L \simeq 20$ kpc as the brown dwarf, its Einstein angular radius as observed from the Earth will be

$$\theta_E \sim \sqrt{\frac{4 \times 6.7 \times 10^{-11} \times 10^{-24}}{(3 \times 10^8)^2} \frac{9 \times 10^{20}}{6 \times 10^{20} \times 15 \times 10^{20}}} \text{ rad} \sim 5 \times 10^{-35} \text{ rad} .$$

This is very much smaller than the angular size of the star. Even if there were a suitable alignment between the lensing object and the background star, the lensing effect would take place locally over a very small fraction of the image of the star. There will not be any measurable increase in the observed flux of light from the star. [Seen as example problem] [6]

(d) Microlensing surveys have detected a number of lensing events towards the Magellanic Clouds, and particularly toward the Galactic Bulge. However, the number of detections is too small for the dark matter halo of the Galaxy to be made entirely of brown dwarfs or low mass stars. [Lectures] [2]

(e) In the monolithic collapse model the Galaxy formed from a large cloud of gas having initially a very low or zero metallicity. The cloud initially collapsed along near-radial paths, forming some stars in this time which in turn enriched the gas with heavy elements. Stars formation during this collapse produced the very low metallicity stars of the Galactic stellar halo, which have elongated orbits that are randomly oriented. These halo stars as a system have little net rotation. The angular momentum of the gas a short time later produced a rotating, flattened gas disc close to hydrostatic equilibrium. Stars that formed from the gas at this time produced the thick disc and were moderately metal-poor, relatively old, were rotationally supported but had some asymmetric drift and appreciable velocity dispersion. The gas settled to a thin disc, which formed stars to produce the stellar disc of the Galaxy. These stars have near-solar metallicity, near-zero asymmetric drift and small velocity dispersion. Gas that fell to the central regions produced the Bulge stars. [Lectures] [5]

(f) Numerical simulations of structure and galaxy formation support the merger model. [Lectures] [1]