ASTM002 / MAS430 The Galaxy

Summer 2004 Examination Paper Model Answers

1. (a) Elliptical and S0 galaxies have relatively red colours. Spiral galaxies have moderately blue colours, with the colour becoming increasingly blue towards later spiral types (Sc and Sd). Irregular galaxies have pronounced blue colours.
[Covered in lectures] [2 marks]

(b) The gas fraction is very small for elliptical and S0 galaxies. For spirals, the gas content increases from type Sa to Sd. Irregulars have a large fraction ($\simeq 15 - 50$ %) of their visible mass in the form of gas. [Lectures] [2]

(c) In collisional encounters interactions between individual particles substantially affect the motions. In collisionless encounters interactions between individual particles do not substantially affect their motions. [Lectures] [3]

(d) Gas is collisional. [Lectures] [2]

(e) Consider a star of mass m approaching a perturbing star of mass m with an impact parameter b. Because the encounter is weak, the change in the direction of motion will be small and the change in velocity will be perpendicular to the initial direction of motion. At any time t when the separation is r, the component of the gravitational force perpendicular to the direction of motion will be

$$F_{perp} = \frac{Gm^2}{r^2}\cos\phi \quad ,$$

where ϕ is the angle at the perturbing mass between the point of closest approach and the perturbed star. [2]

Using $\cos\phi = b/r$ and making the approximation that the speed along the trajectory is constant, $r \simeq \sqrt{b^2 + v^2t^2}$ if $t = 0$ at the point of closest approach, by applying $F = ma$ we obtain

$$\frac{\mathrm{d}v_{perp}}{\mathrm{d}t} = \frac{G\,m\,b}{(b^2 + v^2t^2)^{3/2}} \quad ,$$

where v_{perp} is the component at time t of the velocity perpendicular to the intial direction of motion. [3]

Integrating from time $-\infty$ to ∞,

$$v_{perp} = G\,m\,b \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(b^2 + v^2t^2)^{3/2}} \quad .$$

Using the standard integral in the question, we obtain the final formula. [Lectures] [3]

(f) For the globular cluster, $T_{relax}/T_{cross} \simeq 1.4 \times 10^3$, giving $T_{relax} \simeq 7 \times 10^8$ years. This is much smaller than the age of the globular cluster ($\simeq 14 \times 10^9$ years) and therefore the globular cluster will have relaxed due to two-body encounters over its lifetime. In contrast, for the galaxy, $T_{relax}/T_{cross} \simeq 6 \times 10^8$, giving $T_{relax} \simeq 2 \times 10^{16}$ years. This is much larger than the age of the Universe ($\simeq 14 \times 10^9$ years) and therefore the galaxy will not have relaxed due to two-body encounters over its lifetime. Two body encounters are significant in the globular cluster but not for the galaxy.

[Applying principles discussed in lectures] [6]

(g) Two body encounters can be ignored in modelling the dynamics of galaxies.

[Lectures] [2]

2. (a) Expressing the accelation of a particle in terms of the gradient in the gravitational potential, and the rate of change of position as the velocity,

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial \Phi}{\partial x_i} \qquad \text{and} \qquad \frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i$$

for $i = 1$ to 3. But v_i and x_i are phase space coordinates, so v_i is independent of x_i. Φ is independent of velocity. Therefore,

$$\frac{\partial}{\partial x_i}\left(f\frac{\mathrm{d}x_i}{\mathrm{d}t}\right) = \frac{\partial}{\partial x_i}(f v_i) = v_i\frac{\partial f}{\partial x_i}$$

and

$$\frac{\partial}{\partial v_i}\left(f\frac{\mathrm{d}v_i}{\mathrm{d}t}\right) = -\frac{\partial}{\partial v_i}\left(f\frac{\partial \Phi}{\partial x_i}\right) = -\frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}$$

So

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i\frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i}\right) = 0$$

which is the collisionless Boltzmann equation.
Alternatively, use Hamiltonian dynamics. [Lectures] [6]

(b) From the collisionless Boltzmann equation, f is constant. As the density of stars in space decreases, the density in velocity space must increase, to keep f constant. Therefore the velocity dispersions must decrease.

[Applying material covered in lectures to a new example] [4]

(c) The Galaxy is in a steady state, so $\partial(n\langle v_z\rangle)/\partial t = 0$, and ignoring the negligible terms,

$$\frac{\partial(n\langle v_z^2\rangle)}{\partial z} = -n\frac{\partial \Phi}{\partial z}$$

[2]

If we observe stars towards the Galactic poles (in the z-direction), Poisson's equation reduces to

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G\rho$$

and so

$$\frac{\partial}{\partial z}\left(-\frac{1}{n}\frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right)\right) = 4\pi G\rho$$

[3]

Integrating perpendicular to the galactic plane from $-z$ to z, the surface mass density within a distance z of the plane at the solar Galactocentric radius R_0 is

$$\Sigma(R_0, z) = \int_{-z}^{z}\rho\,\mathrm{d}z' = -\frac{1}{2\pi G n}\frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right)\bigg|_z$$

assuming symmetry about $z = 0$. [Lectures] [3]

(d) If the star densities n can be measured as a function of height z from the plane and the velocities v_z measured as spectroscopic radial velocities, we can solve for $\Sigma(R, z)$ as a function of z. This gives, after modelling the contribution from the dark matter halo, the mass density of the Galactic disc. [Lectures] [5]

(e) No significant dark matter is detected in the Galactic disc as a component of the disc itself. [Lectures] [2]

3. (a) Assumptions behind the Simple Model: (i) the volume of space is a 'closed box' (no gas enters or leaves the volume); (ii) the volume initially contains only unenriched gas (gas initially of zero metallicity and no stars); (iii) the gas is well mixed (the same chemical composition throughout); (iv) instantaneous recycling occurs; (v) the fraction of newly created heavy elements ejected into the gas when gas forms stars is constant. [Lectures] [1 mark for each, up to a max. of 4]

(b) The total mass in the volume is $M_{total} = M_{gas}(t) + M_{stars}(t)$, so a change δM_{gas} in the gas mass produces a change $\delta M_{stars} = -\delta M_{gas}$ in the mass in stars given that $M_{total} = M_{gas}(0)$ is constant. [2]

The change in metallicity is therefore

$$\delta Z \;=\; p\,\frac{\delta M_{stars}}{M_{gas}(0) - M_{stars}(t)} \quad .$$

[Lecturers] [3]

Integrating from time 0 to t, the metallcity at time t is

$$Z(t) \;=\; p\int_0^{M_{stars}(t)} \frac{\mathrm{d}M_{stars}}{M_{gas}(0) - M_{stars}} \;=\; -\,p\,\ln\left(\frac{M_{gas}(0) - M_{stars}(t)}{M_{gas}(0)}\right)$$

[Lectures] [2]

which gives

$$M_{stars}(t) \;=\; M_{gas}(0)\,(1 - \mathrm{e}^{-Z/p})$$

[Lectures] [1]

(c) Dividing the expression for the mass in stars at time t with the expression at the present time gives

$$\frac{M_{stars}(t)}{M_{stars\,1}} \;=\; \frac{1 - \mathrm{e}^{-Z/p}}{1 - \mathrm{e}^{-Z_1/p}}$$

[Lectures] [1]

If we observe a subsample of long-lived stars, each of mass m, the numbers of these stars at time t will be kM_{stars} where k is a constant for a constant initial mass function. Therefore the number of stars with a metallicity less than Z will be $N(Z) = kM_{stars}$. The number having a metallicity less than the current value is $N_1 = kM_{stars\,1}$. Dividing these,

$$\frac{N(Z)}{N_1} \;=\; \frac{1 - \mathrm{e}^{-Z/p}}{1 - \mathrm{e}^{-Z_1/p}}$$

[Lectures] [4]

(d) The Simple Model predicts far more low metallicity stars than are observed. This is known as the *G dwarf problem*. [Lectures] [4]

(e) Pre-enrichment of the gas would reduce the predicted numbers of low-metallicity stars and would lessen the discrepancy between model and observations. [Lectures] [2]

(f) Losing enriched gas from the interstellar medium would skew the metallicity distribution to lower heavy element abundances. [Lectures] [2]

4. (a) Optical spectra are restricted to a limited radial distance from the centre of the galaxy, to typically a few scale lengths. The gravitational effect of an exponential spiral disc produces a rising rotation curve out to $2-3$ scale lengths. Optical rotation curves typically do not reach a large enough distance from the centre to allow the signature of a dark halo to be inferred unambiguously. [Lectures] [2]

(b) 21cm radio rotation curves can be observed to considerably greater distance from the centre of the galaxy than optical curves. They extend well beyond the point at $2-3$ optical scale lengths where the peak in the circular velocity of a pure exponential disc is found. [Lectures] [2]

(c) The 21cm line is produced by hyperfine transitions in atomic hydrogen between parallel and anti-parallel angular momentum states, caused by the coupling of the angular momenta of the proton and electron. [Lectures] [2]

(d) The transition probability is so low that interactions between atoms interfere before appreciable emission / absorption from the 21cm transition can occur in the laboratory. [Lectures] [1]

(e) The optical depth of microlenses is the fraction of the solid angle covered by the Einstein rings of the lensing objects. [Lectures] [3]

(f) Consider a field subtending a solid ange Ω. The fraction of this field covered by Einstein radii of lensing sources at distances between D_L and $D_L + \mathrm{d}D_L$ is $\mathrm{d}\tau = \pi\, \theta_E^2\, \mathrm{d}N \,/\, \Omega$ where $\mathrm{d}N$ is the number of lenses in this thin volume. If n is the number density of lenses, $\mathrm{d}N = n\, D_L^2\, \mathrm{d}D_L\, \Omega$. The mass density is $\rho = nM_L$. Therefore,

$$\mathrm{d}\tau \;=\; \frac{\pi\, \theta_E^2\, \rho\, D_L^2\, \mathrm{d}D_L}{M_L} \quad .$$

Substituting for the angular Einstein radius and integrating over distance, we obtain the required result. [Lectures] [7]

(g) The distance of a lens from the Galactic Centre is $r = \sqrt{D_L^2 + R_0^2}$ while the lens-source distance is $D_{LS} = R_0 - D_L$. Substituting for $\rho(r)$ into the expression for the optical depth,

$$\begin{aligned}
\tau &= \frac{2\,k}{c^2 D_S} \int_0^{R_0} \frac{D_L\,(R_0 - D_L)}{D_L^2 + R_0^2}\, \mathrm{d}D_L \\
&= \frac{2\,k}{c^2 D_S} \int_0^{R_0} \left(-1 \,+\, \frac{R_0^2}{D_L^2 + R_0^2} \,+\, \frac{R_0\, D_L}{D_L^2 + R_0^2} \right) \mathrm{d}D_L \\
&= \frac{2\,k}{c^2 D_S} \left[-D_L \,+\, R_0\, \tan^{-1}\left(\frac{D_L}{R_0}\right) \,+\, \frac{R_0}{2} \ln(R_0^2 + D_L^2) \right]_0^{R_0} \\
&= \frac{kR_0}{2c^2 D_S}\, (\, 2 \ln 2 \,+\, \pi \,-\, 4\,)
\end{aligned}$$

$$= \quad \frac{k}{2c^2}\,(\,2\ln 2 \;+\; \pi \;-\; 4\,) \quad .$$

[New application of principles discussed in lectures] [7]

(h) The optical depth for microlensing by compact objects comprising the dark halo is expected to be very small ($\sim 10^{-7}$). Therefore very large numbers of stars $> 10^6$ must be monitored to give reasonable chances of detecting of microlensing events, requiring fields with $> 10^6$ stars. [Applying principles discussed in lectures] [2]