Appendix D: Example Problems

Problem 20: Answer

The metallicity of the interstellar medium is $Z = -p \ln \mu$ in the Simple Model, where p is the yield (a constant) and μ us the gas fraction (the fraction of the total baryonic mass in the form of gas) at any time. As $\mu \rightarrow 0$, $Z \rightarrow \infty$ formally.
This is not physically realistic. From the definition of the heavy element mass fraction Z, material consisting entirely of heavy elements (no hydrogen or helium) would have $Z = 1$, the maximum value Z can take. However, this $Z = -p \ln \mu$ prediction only gives $Z \simeq 1$ for extremely small gas fractions ($\mu \sim 10^{-22}$) which are not likely to be encountered in any real galaxy.
The formal failure of the Simple Model prediction is the result of the approximations made assuming Z is small. In practice, $Z \ll 1$ always and the Simple Model equations work accurately, provided that the basic assumptions behind the Simple Model are valid. (If we rederived the equations without these small Z approximations, the Simple Model would give $Z \rightarrow 1$ as $\mu \rightarrow 0$.)

The gas fraction $\mu = M_{\rm gas}/M_{\rm gas}(0)$. But $M_{\rm stars} + M_{\rm gas} = M_{\rm total} = M_{\rm gas}(0)$, which gives, $M_{\rm gas} = M_{\rm gas}(0) - M_{\rm stars}$. Therefore, the gas fraction is

$$\mu = \frac{M_{\rm gas}(0) - M_{\rm stars}}{M_{\rm gas}(0)} = 1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)} . \qquad \text{(A)}$$

The $Z = -p \ln \mu$ result can therefore be rewritten as

$$Z = -p \ln \left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)} \right) .$$

The mean metallicity is $\langle Z \rangle = \dfrac{\int_0^{M_{\rm stars}} Z \, {\rm d}M'_{\rm stars}}{\int_0^{M_{\rm stars}} {\rm d}M'_{\rm stars}}$. Therefore,

$$\begin{aligned}
\langle Z \rangle &= \frac{1}{M_{\rm stars}} \int_0^{M_{\rm stars}} Z \, {\rm d}M'_{\rm stars} = \frac{1}{M_{\rm stars}} \int_0^{M_{\rm stars}} -p \ln \left(1 - \frac{M'_{\rm stars}}{M_{\rm gas}(0)} \right) {\rm d}M'_{\rm stars} \\
&= -\frac{p}{M_{\rm stars}} \left[\left(M'_{\rm stars} - M_{\rm gas}(0) \right) \ln \left(1 - \frac{M'_{\rm stars}}{M_{\rm gas}(0)} \right) - M'_{\rm stars} \right]_0^{M_{\rm stars}} \\
&= -\frac{p}{M_{\rm stars}} \left(\left(M_{\rm stars} - M_{\rm gas}(0) \right) \ln \left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)} \right) - M_{\rm stars} - 0 \right) \\
&= p \left(\left(-1 + \frac{M_{\rm gas}(0)}{M_{\rm stars}} \right) \ln \left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)} \right) + 1 \right) ,
\end{aligned}$$

which gives the required answer,

$$\langle Z \rangle = p \left(\frac{M_{\rm gas}(0)}{M_{\rm stars}} - 1 \right) \ln \left(1 - \frac{M_{\rm stars}}{M_{\rm gas}(0)} \right) + p .$$

Rearranging equation (a),

$$\frac{M_{\rm stars}}{M_{\rm gas}(0)} = 1 - \mu . \qquad \therefore \quad \frac{M_{\rm gas}(0)}{M_{\rm stars}} = \frac{1}{1 - \mu} . \qquad \text{(B)}$$

Substituting from (A) and (B),

$$\langle Z\rangle \;=\; p\left(\frac{1}{1-\mu}-1\right)\ln\mu \;+\; p \;=\; p\left(\frac{1-(1-\mu)}{1-\mu}\right)\ln\mu \;+\; p \;=\; p\left(1+\frac{\mu\ln\mu}{1-\mu}\right) \;,$$

the required result.

As gas is exhausted in star formation, $\mu \to 0$, $\mu\ln\mu/(1-\mu) \to 0$. Therefore, $\langle Z\rangle \to p$, the yield. (This is an interesting and important prediction of the Simple Model. In practice, $\mu > 0$ and therefore we expect the mean metallicity in a population of long-lived stars to be less than the yield.)

Problem 21: Answer

We can apply the same analysis to this problem as was used for the Simple Model, but the total mass $M_{\rm total}$ is now a function of time t.
The total change in the total mass between time 0 and t is $\Delta M_{\rm total}(t) = M_{\rm total}(t) - M_{\rm total}(0)$, and the total change in the mass in stars is $\Delta M_{\rm stars}(t) = M_{\rm stars}(t) - M_{\rm stars}(0) = M_{\rm stars}(t)$ because there were no stars initially. Since $\delta M_{\rm total} = -\,c\,\delta M_{\rm stars}$, $\Delta M_{\rm total} = -\,c\,\Delta M_{\rm stars}$, and so, $M_{\rm total}(t) - M_{\rm total}(0) = -\,c\,M_{\rm stars}(t)$. Since $M_{\rm total}(t) = M_{\rm stars}(t) + M_{\rm gas}(t)$ and the mass of gas at time t is

$$\begin{aligned} M_{\rm gas}(t) \;&=\; M_{\rm total}(t) \;-\; M_{\rm stars}(t) \;=\; M_{\rm total}(0) \;-\; c\,M_{\rm stars}(t) \;-\; M_{\rm stars}(t) \\ &=\; M_{\rm total}(0) \;-\; (1+c)\,M_{\rm stars}(t) \;\;. \end{aligned}$$

Some results from the Simple Model still apply, such as the expression for the change δZ in the metallicity Z of the gas in time δt, and the relation between the changes in the mass in stars and the total mass $M_{\rm SF}$ that has participated in star formation up to time t:

$$\delta Z \;=\; \frac{\delta M_{\rm metals}}{M_{\rm gas}} \;-\; Z\,\frac{\delta M_{\rm gas}}{M_{\rm gas}} \qquad \text{and} \qquad \delta M_{\rm stars} \;=\; \alpha\,\delta M_{\rm SF} \;\;,$$

where α is the fraction of the mass participating in star formation that remains in long-lived stars and stellar remnants. With outflow we have

$$\begin{aligned} \delta M_{\rm metals} \;&=\; -\,Z\,\delta M_{\rm SF} \;+\; Z\,(1-\alpha)\,\delta M_{\rm SF} \;+\; p\,\delta M_{\rm stars} \;-\; c\,Z\delta M_{\rm stars} \\ &=\; p\,\delta M_{\rm stars} \;-\; Z\,\delta M_{\rm stars} \;-\; c\,Z\delta M_{\rm stars} \;\;. \end{aligned}$$

Substituting in the expression for δZ gives the required result

$$\delta Z \;=\; \frac{p\,\delta M_{\rm stars}}{M_{\rm total}(0) \;-\; (1+c)\,M_{\rm stars}} \;\;.$$

But this is just the closed-box result with p replaced by $p/(1+c)$, and $M_{\rm total} = M_{\rm gas}(0)$ replaced by $M_{\rm total}(0)/(1+c)$. So the leaky box model just changes the effective value of p and doesn't change the distribution of stellar metallicities.

Appendix D: Example Problems

Problem 22: Answer

The angular size corresponding to the Einstein radius is (from the course notes)

$$\theta_E \;=\; \sqrt{\frac{4GM}{c^2}\,\frac{D_{LS}}{D_L D_S}} \quad ,$$

where M is the mass of the lensing object, D_L is the distance to the lensing object, D_S is the distance to the source, D_{LS} is the distance between the lens and source, G is the constant of gravitation and c is the speed of light.
For $D_S \gg D_L$, we have $D_{LS} \simeq D_S$.

$$\therefore \quad \theta_E \;=\; \frac{2\sqrt{G}}{c}\sqrt{\frac{M}{D_L}} \;\; \text{rad} \;=\; \frac{2\sqrt{6.673\times 10^{-11}}}{2.998\times 10^{8}}\sqrt{\frac{M}{D_L}} \;\; \text{rad kg}^{-1/2}\,\text{m}^{1/2}$$

$$=\; 5.45\times 10^{-14}\sqrt{\frac{M}{D_L}} \;\; \text{rad kg}^{-1/2}\,\text{m}^{1/2}$$

$$=\; 5.45\times 10^{-14}\sqrt{\frac{M}{D_L}}\left(\frac{180\times 60\times 60}{\pi}\right)\left(\frac{1.989\times 10^{30}}{3.086\times 10^{16}}\right)^{1/2} \;\; \text{arcsec}\; M_\odot^{-1/2}\,\text{pc}^{1/2}$$

$$=\; 0.081\sqrt{\frac{M}{D_L}} \;\; \text{arcsec}\; M_\odot^{-1/2}\,\text{pc}^{1/2}$$

So the constant is $k = 0.081$ arcsec $M_\odot^{-1/2}$ pc$^{1/2}$.
This means that a lens at a distance of 1 pc (about the distance of the nearest star) imaging an extragalactic source will produce an Einstein angular radius of about 0.1 arcsec.

Problem 23: Answer

Use the equation

$$\tau \;=\; \frac{4\pi G}{c^2 D_{\rm S}}\int_0^{D_{\rm S}} D_{\rm L}\, D_{\rm LS}\, \rho(D_{\rm L})\, {\rm d}D_{\rm L} \quad ,$$

where D_S is the distance from the observer to the source, D_L is the distance from the observer to the lens, and D_{LS} is the distance from the lens to the source. Considering the geometry, $D_S = R_0$, $D_{LS} = R_0 - D_L$ and $D_{LS} = r$. Therefore, $r = R_0 - D_L$ and ${\rm d}D_L = -{\rm d}r$. The optical depth is then

$$\tau \;=\; \frac{4\pi G}{c^2 R_0}\int_0^{R_0}(R_0 - r)\, r\,\frac{\rho_0 b^2}{r^2+b^2}\,{\rm d}r \;=\; \frac{4\pi G\rho_0 b^2}{c^2 R_0}\int_0^{R_0}\left(\frac{R_0\, r}{r^2+b^2} \;-\; \frac{r^2}{r^2+b^2}\right){\rm d}r$$

$$=\; \frac{4\pi G\rho_0 b^2}{c^2 R_0}\left[\frac{R_0}{2}\,\ln(r^2+b^2) \;-\; r \;+\; b\,\tan^{-1}\left(\frac{r}{b}\right)\right]_0^{R_0}$$

using the standard integral $\int r^2/(r^2+b^2)\,{\rm d}r \;=\; r \;-\; b\tan^{-1}(r/b) + \text{constant}$, given in the question. This gives,

$$\tau \;=\; \frac{2\pi G\rho_0 b^2}{c^2}\left(\ln\left(1+\frac{R_0^2}{b^2}\right) \;-\; 2 \;+\; \frac{2b}{R_0}\tan^{-1}\frac{R_0}{b}\right)$$

Substituting for $\rho_0 = 2.0 \times 10^{-20}\,\mathrm{kg\,m^{-3}}$, $b = 2.0 \times 10^3$ pc $= 2.0 \times 10^3 \times 3.086 \times 10^{16}$m, $R_0 = 8.0 \times 10^3$ pc $= 8.0 \times 10^3 \times 3.086 \times 10^{16}$m, gives $\tau = 5.3 \times 10^{-7}$.

Problem 24: Answer

The expression for the Einstein angular radius is

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{\mathrm{LS}}}{D_{\mathrm{L}}D_{\mathrm{S}}}} \text{ rad} ,$$

where the mass of the lens M is $1000\,m_P = 1.673 \times 10^{-24}$ kg, the distance between the observer and lens is $D_L = 20$ kpc $= 6.17 \times 10^{20}$ m, the distance between the observer and source is $D_S = 50$ kpc $= 1.54 \times 10^{21}$ m, and the distance between the lens and source is $D_{LS} = 30$ kpc $= 9.26 \times 10^{20}$ m. For the WIMP we have $\theta_E = 2.2 \times 10^{-36}$ rad. The star with a radius 6.96×10^8 m at a distance of 50 kpc $= 1.54 \times 10^{21}$ m subtends an angular radius of $6.96 \times 10^8 / 1.54 \times 10^{21}$ rad $= 4.5 \times 10^{-13}$ rad. So the angular radius of the star is 5×10^{23} times the Einstein angular radius of the WIMP. The lensing effect of the WIMP will take place on a scale that is $\sim 10^{-23}$ smaller than the scale of the star image. The lensing effects will therefore occur only within a negligibly small region of the image of the star. The change in the light received from the star will be negligible. It will be completely undetectable.

[This question assumes that WIMPs do exist. Despite dedicated experiments to search for them, they have not yet been detected yet.]

A $M = 0.05M_\odot$ brown dwarf will have $\theta_E = 5.4 \times 10^{-10}$ rad $= 1.1 \times 10^{-4}$ arcsec. The Einstein angular radius of the brown dwarf is 1190 times larger than that angular radius of the background star. The image of the background star will have its solid angle appreciably changed if it is lensed by the brown dwarf. Lensing effects will be significant if there is a suitable alignment.

Appendix D: Example Problems

Problem 25: Answer

The formulae can be shown to be solutions by differentiation and appropriate substitution into the differential equation.
To prove the sinh and cosh equations are a solution, differentiating $l = (GM\tau_0^{\,2})^{1/3}(\cosh\eta - 1)$ and $t = \tau_0\,(\sinh\eta - \eta)$, we get

$$\frac{\mathrm{d}l}{\mathrm{d}\eta} = (GM\tau_0^{\,2})^{1/3}\sinh\eta \qquad \text{and} \qquad \frac{\mathrm{d}t}{\mathrm{d}\eta} = \tau_0\,(\cosh\eta - 1)\ .$$

Therefore, $\dfrac{\mathrm{d}l}{\mathrm{d}t} = \dfrac{\mathrm{d}l}{\mathrm{d}\eta}\left(\dfrac{\mathrm{d}t}{\mathrm{d}\eta}\right)^{-1} = \dfrac{(GM\tau_0^{\,2})^{1/3}\sinh\eta}{\tau_0\,(\cosh\eta - 1)}$. Differentiating again,

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\mathrm{d}l}{\mathrm{d}t}\right) = \frac{(GM\tau_0^{\,2})^{1/3}\cosh\eta\,\tau_0\,(\cosh\eta - 1) - (GM\tau_0^{\,2})^{1/3}\sinh\eta\,\tau_0\sinh\eta}{\tau_0^{\,2}(\cosh\eta - 1)^2}$$

$$= -\frac{(GM\tau_0^{\,2})^{1/3}}{\tau_0\,(\cosh\eta - 1)} \qquad \text{using} \quad \cosh^2 x - \sinh^2 x \equiv 1\ .$$

Therefore,

$$\frac{\mathrm{d}^2 l}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\mathrm{d}l}{\mathrm{d}t}\right).\left(\frac{\mathrm{d}t}{\mathrm{d}\eta}\right)^{-1} = -\frac{(GM\tau_0^{\,2})^{1/3}}{\tau_0(\cosh\eta - 1)}\frac{1}{\tau_0(\cosh\eta - 1)} = -\frac{(GM\tau_0^{\,2})^{1/3}}{\tau_0^{\,2}(\cosh\eta - 1)^2}$$

Therefore,

$$\frac{\mathrm{d}^2 l}{\mathrm{d}t^2} + \frac{GM}{l^2} = -\frac{(GM\tau_0^{\,2})^{1/3}}{\tau_0^{\,2}(\cosh\eta - 1)^2} + \frac{GM}{[(GM\tau_0^{\,2})^{1/3}(\cosh\eta - 1)]^2}$$

$$= -\frac{(GM)^{1/3}}{\tau_0^{\,4/3}(\cosh\eta - 1)^2} + \frac{(GM)^{1/3}}{\tau_0^{\,4/3}(\cosh\eta - 1)^2} = 0\ .$$

So, $\dfrac{\mathrm{d}^2 l}{\mathrm{d}t^2} = -\dfrac{GM}{l^2}$, the original equation of motion.
Therefore the sinh and cosh parametric equations are solutions to the equation of motion.

To prove the power law equation is a solution, differentiate $l = kt^n$ to get $\mathrm{d}l/\mathrm{d}t = nkt^{n-1}$, and again to get $\mathrm{d}^2l/\mathrm{d}t^2 = n(n-1)kt^{n-2}$. Substituting for l and $\mathrm{d}^2l/\mathrm{d}t^2$ into the equation of motion, we get

$$n(n-1)\,k\,t^{n-2} = -\frac{GM}{k^2 t^{2n}} = -\frac{GM}{k^2}\,t^{-2n}$$

Note that this equality must be true for all values of t (not at just a restricted number of values). This requires the indices of t to be equal. So $n - 2 = -2n$, so $n = 2/3$. The coefficients must also be equal. Therefore,

$$n(n-1)k = -\frac{GM}{k^2}\,, \qquad \text{and so,}\ k^3 = -\frac{GM}{n(n-1)} = -\frac{GM}{\frac{2}{3}(\frac{2}{3} - 1)} = \frac{9GM}{2}$$

This gives $k = \left(\frac{9GM}{2}\right)^{1/3}$. So the solution is $l = \left(\frac{9GM}{2}\right)^{1/3} t^{2/3}$.

The sinh and cosh solution represents the situation where the mass of the Galaxy–M31 system is insufficient to reverse their initial movement apart. They continue to move apart at all times, with $\mathrm{d}l/\mathrm{d}t > 0$ even in the limit $t \to \infty$. The separation l continues to increase with time at the present day in this solution: the solution predicts that M31 will still be moving away from the Galaxy today, which is inconsistent with radial velocity measurements which show that they they are now moving together.

The power law solution represents the situation where the mass of the Galaxy–M31 system is just insufficient to reverse their movement apart. The separation l continues with time to the present day, but $\mathrm{d}l/\mathrm{d}t \to 0$ in the limit $t \to \infty$. It predicts that M31 will still be moving away from the Galaxy today, which is inconsistent with radial velocity measurements.

Problem 26: Answer

Expanding $\sigma_\phi^2 \equiv \langle\, (\, v_\phi \,-\, \langle v_\phi\rangle\,)^2\,\rangle$,

$$\begin{aligned}\sigma_\phi^2 &= \langle\, v_\phi^2 - 2v_\phi\langle v_\phi\rangle + \langle v_\phi\rangle^2\,\rangle \;=\; \langle v_\phi^2\rangle - \langle 2v_\phi\langle v_\phi\rangle\rangle + \langle\langle v_\phi\rangle^2\rangle \\ &= \langle v_\phi^2\rangle - 2\,\langle v_\phi\rangle\,\langle v_\phi\rangle + \langle v_\phi\rangle^2 \;=\; \langle v_\phi^2\rangle - 2\,\langle v_\phi\rangle^2 + \langle v_\phi\rangle^2 \;=\; \langle v_\phi^2\rangle - \langle v_\phi\rangle^2\end{aligned}$$

So $\sigma_\phi^2 = \langle v_\phi^2\rangle \,-\, \langle v_\phi\rangle^2$ is the required relation between σ_ϕ, $\langle v_\phi^2\rangle$ and $\langle v_\phi\rangle$.

By a similar analysis, $\sigma_R^2 = \langle v_R^2\rangle - \langle v_R\rangle^2$ and $\sigma_z^2 = \langle v_z^2\rangle - \langle v_z\rangle^2$. However, the Galaxy is in a steady state, so $\langle v_R\rangle = 0$ and $\langle v_z\rangle = 0$. Therefore,

$$\sigma_R^2 \;=\; \langle v_R^2\rangle \quad \text{and} \quad \sigma_z^2 \;=\; \langle v_z^2\rangle\ .$$

The difference between v_{circ}^2 and $\langle v_\phi\rangle^2$ at any point in the Galaxy is

$$v_{circ}^2 - \langle v_\phi\rangle^2 \;=\; (v_{circ} - \langle v_\phi\rangle)\,(v_{circ} + \langle v_\phi\rangle)\ ,$$

using the general mathematical result $a^2 - b^2 \equiv (a+b)(a-b)$. The asymmetric drift is $v_a \equiv v_{circ} - \langle v_\phi\rangle$ by definition. Therefore,

$$v_{circ}^2 - \langle v_\phi\rangle^2 \;=\; v_a\,(v_{circ} + \langle v_\phi\rangle) \;=\; v_a\,(v_{circ} + (v_{circ} - v_a)) \;=\; v_a\,(2v_{circ} - v_a)\ ,$$

the required result.

For stars in the Galactic disc, the asymmetric drift is small with $v_a \ll 2v_{circ}$. Therefore, $(2v_{circ} - v_a) \simeq 2v_{circ}$. This gives

$$v_a \;\simeq\; \frac{v_{circ}^2 - \langle v_\phi\rangle^2}{2v_{circ}}$$

on rearranging, the required result.

Substituting for $v_{circ}^2 - \langle v_\phi\rangle^2$ from the expression $v_{circ}^2 - \langle v_\phi\rangle^2 \;=\; -\,F\,\langle v_R^2\rangle$, and using $\langle v_R^2\rangle = \sigma_R^2$ (derived above),

$$v_a \;\simeq\; -\,F\,\frac{\sigma_R^2}{2v_{circ}}\ .$$

So $v_a \propto \sigma_R^2$ approximately. [Note that the asymmetric drift v_a is negative.]