Appendix D: Example Problems

Problem 14: Answer

Because there is no net rotation and the velocity dispersion σ is constant and isotropic, $\langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle = \sigma^2$. The Jeans equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n\,\sigma^2\right) + \frac{n}{r}\,(0) = -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} \qquad \therefore \quad \sigma^2\,\frac{\mathrm{d}n}{\mathrm{d}r} = -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

Integrating,

$$\sigma^2 \int_{n_0}^{n(r)} \frac{\mathrm{d}n}{n} = -\int_{\Phi(0)}^{\Phi(r)} \mathrm{d}\Phi' \,.$$

$$\therefore \quad \sigma^2 \Big[\ln n\Big]_{n_0}^{n(r)} = -\,\Phi(r) + \Phi(0)$$

$$\therefore \quad \sigma^2\left(\ln n(r) - \ln n_0\right) = \frac{GM_{tot}}{(r^2+a^2)^{1/2}} - \frac{GM_{tot}}{a}$$

$$\sigma^2 \ln\left(\frac{n(r)}{n_0}\right) = \frac{GM_{tot}}{a}\left(\frac{1}{\sqrt{1+r^2/a^2}} - 1\right),$$

which gives the required result.

[This is a more direct approach than differentiating the expression for Φ to substitute for $\mathrm{d}\Phi/\mathrm{d}r$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n\,\sigma^2\right) + \frac{n}{r}\,(0) = -\,n\,\frac{GM_{tot}\,r}{(r^2+a^2)^{3/2}}, \qquad \therefore \quad \sigma^2\,\frac{\mathrm{d}n}{\mathrm{d}r} = -\,n\,\frac{GM_{tot}\,r}{(r^2+a^2)^{3/2}},$$

Integrating,

$$\sigma^2 \int_{n_0}^{n(r)} \frac{\mathrm{d}n}{n} = -\,GM_{tot}\int_0^r \frac{r'}{(r'^2+a^2)^{3/2}}\mathrm{d}r' \,.$$

$$\therefore \quad \sigma^2 \Big[\ln n\Big]_{n_0}^{n(r)} = -\,GM_{tot}\left[-\frac{1}{(r'^2+a^2)^{1/2}}\right]_0^r .$$

$$\therefore \quad \sigma^2\left(\ln n(r) - \ln n_0\right) = GM_{tot}\left(\frac{1}{(r^2+a^2)^{1/2}} - \frac{1}{a}\right)$$

which gives the required result.]

Problem 15: Answer

One Balmer photon is produced by the interstellar gas from each Lyman continuum photon from the stars inside the H II region (because a large majority of the H atoms in the gas are in the ground state). Summing the Balmer emission, the Balmer flux from the H II region is 2.7×10^7 photons $\mathrm{s^{-1}\,m^{-2}}$. The H II region is at a distance of 900 pc. Therefore the Balmer luminosity from it is dispersed over the area of a sphere of radius 900 pc when it reaches the Earth. The Balmer luminosity is therefore $2.7 \times 10^7 \times 4\pi\,(900 \times 3.0857 \times 10^{16})^2$ photons $\mathrm{s^{-1}} = 2.6 \times 10^{47}$ photons $\mathrm{s^{-1}}$. Therefore the total luminosity of ultraviolet photons with wavelengths shorter than 912 Å from stars inside the H II region is 2.6×10^{47} photons s^{-1}.

[The wavelength 912 Å is significant because a photon of this wavelength has an energy of 13.6 eV – the ionisation energy of hydrogen atoms – and can therefore ionise hydrogen atoms in the ground state.]

Problem 16: Answer

The ideal gas law relates the pressure P in a gas to the number density n of particles and the absolute temperature T by $P = n\, k_B\, T$, where k_B is the Boltzmann constant. This gives the pressure in the hot ionised gas as $P_{hot} = 6000\ \mathrm{m}^{-3} \times k_B \times 500\,000\ \mathrm{K} = 3 \times 10^9\, \mathrm{k_B}\ \mathrm{K\,m^{-3}}$ (in this problem, it is easier to leave the pressure in terms of the constant k_B than to calculate the result explicitly: we only need to compare figures and are not bothered about the actual values). For the cold neutral gas we find that the pressure is $P_{cold} = 1 \times 10^9\, k_B\ \mathrm{K\,m^{-3}}$. For the warm neutral gas the pressure is $P_{warm} = 1 \times 10^9\, k_B\ \mathrm{K\,m^{-3}}$.
We can see that $P_{cold} \simeq P_{warm} \neq P_{hot}$. Therefore the cold neutral region and the warm neutral region are in pressure equilibrium. The hot ionised region is not in equilibrium with the other two regions.

[Because the hot ionised region has a higher pressure than the others, it will expand by compressing them, although the higher densities in the others mean that their inertias slow the process significantly.]
[In practice, magnetic fields and cosmic rays can provide contributions to the pressure in addition to the gas pressure considered here. They will increase the pressures, particularly for the hot ionised gas which will have been produced by supernovae (supernovae will produce cosmic rays and the neutron stars they leave behind can add to the strength of the magnetic field). These extra effects are ignored in this case.]

Appendix D: Example Problems

Problem 17: Answer

The $(B-V)$ colour excess of the star is $E_{(B-V)} = (B-V) - (B-V)_0 = 0.98 - 0.76 = 0.22$ mag. The mean interstellar extinction curve has the relation $A_V = 3.1 E_{(B-V)}$. Therefore, we expect $A_V = 3.1 \times 0.22 = 0.68$ mag. The relation between apparent and absolute magnitude for the V band is $V - M_V = 5\log_{10}(D/\mathrm{pc}) - 5 + A_V$, a standard formula (given in Appendix A of the course notes). Therefore, the distance is $D = 10^{(V-M_V+5-A_V)/5}$ pc $= 10^{13.60-5.20+5-0.68}$ pc $= 350$ pc. The distance to the star is 350 pc.

The mean extinction per unit distance is 0.73 mag/350 pc = 0.73 mag/0.350 kpc = 1.95 mag $(\mathrm{kpc})^{-1}$.

[Note the linear relation between extinction and distance. This comes from the extinction $A_V = 1.086\tau_V$ where τ_V is the optical depth in the V band. Put $\tau_V = \rho_d\,\kappa_V\,D$, where ρ_d is the density of dust in space, κ_V is a coefficient expressing how strongly dust absorbs light in the V band (a constant for the V band unless the type of dust particles varies substantially), and D is the distance to the star. So $A_V = 1.086\,\rho_d\,\kappa_V\,D$ and therefore A_V varies linearly with distance D. This means that it makes sense to use the extinction per unit distance.]

This figure depends on the density of dust in space. Therefore it will be large (as in this case) in the direction of the Milky Way, and small away from the plane of the Galaxy. It will be large along sight lines that pass through dense gas (such as cold, neutral gas) and smaller along sight lines through lower density gas (such as hot ionised gas). Therefore the mean extinction per unit distance varies strongly across the sky.

[In the $A_V = 1.086\,\rho_d\,\kappa_V\,D$ representation used above, $A_V/D = 1.086\,\rho_d\,\kappa_V$. The parameter κ_V will vary only slightly, but ρ_d varies greatly. Therefore, A_V/D varies greatly.]

To estimate the extinction in the I and K photometric bands, we need information about how the extinction in those bands compares to the extinction or reddening in the optical, because the question only gives us information about the optical part of the spectrum. From the mean interstellar extinction law graph (in Section **??** of the course notes), $A_I/E_{(B-V)} = 2.0$, and $A_K/E_{(B-V)} = 0.4$ by extrapolation (the K band at 2.2μm = 2200 nm lies just off the edge of the figure). So $A_I = 2.0E_{(B-V)} = 2.0 \times 0.22 = 0.44$mag, and $A_K = 0.4E_{(B-V)} = 0.4 \times 0.22 = 0.09$mag. So the extinction in the I and K bands will be 0.44 and 0.09 mag respectively.

Problem 18: Answer

The surface mass density of stars can be obtained by integrating the density over height z:

$$\Sigma_s = \int_{-\infty}^{\infty} \rho_s(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} \rho_{so}\,\mathrm{e}^{-|z|/h_s}\,\mathrm{d}z = 2\int_0^{\infty} \rho_{so}\,\mathrm{e}^{-z/h_s}\,\mathrm{d}z \quad \text{(from symmetry)}$$

$$= 2\,\rho_{so}\int_0^{\infty} \mathrm{e}^{-z/h_s}\,\mathrm{d}z = 2\,\rho_{so}\,\left[-h_s\,\mathrm{e}^{-z/h_s}\right]_{z=0}^{\infty} = 2\,\rho_{so}\,h_s$$

Similarly, for the gas, $\Sigma_g = 2\,\rho_{go}\,h_g$. Therefore,

$$\frac{\Sigma_s}{\Sigma_g} = \frac{2\,\rho_{so}\,h_s}{2\,\rho_{go}\,h_g} = \frac{\rho_{so}}{\rho_{go}}\,\frac{h_s}{h_g} = 6 \times \frac{250}{150} = 10 \ .$$

So $\Sigma_s = 10\,\Sigma_g$ at the Sun's distance from the Galactic Centre.
Dust density ρ_d closely follows that of gas and observations show that $\rho_d/\rho_g \simeq 0.1$. Therefore we expect $\Sigma_s \simeq 100\,\Sigma_d$ at the Sun's distance from the Galactic Centre.
[In reality, the density of the interstellar medium is highly variable from place to place. This exponential law is a mean representation of the decline in density from the Galactic plane.]

Problem 19: Answer

The observed colour indices of an object that is affected by reddening by dust are (U – B) and (B – V). The intrinsic values (what would be observed if no dust were present) are $(\mathrm{U-B})_0$ and $(\mathrm{B-V})_0$. The colour excesses (which measure the extent of the reddening by dust) are therefore $E_{\mathrm{U-B}} = (\mathrm{U-B}) - (\mathrm{U-B})_0$ and $E_{\mathrm{B-V}} = (\mathrm{B-V}) - (\mathrm{B-V})_0$. So the observed values are $(\mathrm{U-B}) = (\mathrm{U-B})_0 + E_{\mathrm{U-B}}$ and $(\mathrm{B-V}) = (\mathrm{B-V})_0 + E_{\mathrm{B-V}}$. Substituting for these into the expression $Q \equiv (\mathrm{U-B}) - (E_{\mathrm{U-B}}/E_{\mathrm{B-V}})(\mathrm{B-V})$,

$$\begin{aligned} Q &= (\mathrm{U-B})_0 + E_{\mathrm{U-B}} - \frac{E_{\mathrm{U-B}}}{E_{\mathrm{B-V}}}\left((\mathrm{B-V})_0 + E_{\mathrm{B-V}}\right) \\ &= (\mathrm{U-B})_0 + E_{\mathrm{U-B}} - \frac{E_{\mathrm{U-B}}}{E_{\mathrm{B-V}}}(\mathrm{B-V})_0 + E_{\mathrm{U-B}} \\ &= (\mathrm{U-B})_0 - \frac{E_{\mathrm{U-B}}}{E_{\mathrm{B-V}}}(\mathrm{B-V})_0 \end{aligned}$$

But $E_{\mathrm{U-B}}/E_{\mathrm{B-V}}$ is independent of extinction. So the Q parameter is equal to the value it would have in the absence of interstellar extinction. So the value of Q does not depend on the strength of the extinction.
This is the proof required.

Putting the magnitudes of the hot star and $E_{\mathrm{U-B}}/E_{\mathrm{B-V}} = 0.72$ into the expression for Q, we get $Q = -0.84$ mag. As we have established, Q is independent of reddening, so the intrinsic value of Q is -0.84 also. From the table, $Q = -0.84$ corresponds to a spectral type B0V.
The observed (B – V) colour index is $(\mathrm{B-V}) = 12.69 - 12.00 = 0.69$ mag. From the table, the intrinsic (B – V) index for this type of the star is $(\mathrm{B-V})_0 = -0.31$ mag. The (B – V) colour excess is therefore $E_{\mathrm{B-V}} = (\mathrm{B-V}) - (\mathrm{B-V})_0 = 0.69 - (-0.31) = 1.00$ mag.
The V-band extinction can be calculated using the standard relation $A_V = 3.1 E_{\mathrm{B-V}}$. We get $A_V = 3.1$ mag.
The standard formula $m - M = 5\log_{10}(D/\mathrm{pc}) - 5 + A$ provides the relationship between the observed apparent magnitude m, absolute magnitude M, distance D and extinction A (see Appendix A). For the V band we get

$$\mathrm{V} - M_{\mathrm{V}} = 5\log_{10}(D/\mathrm{pc}) - 5 + A_{\mathrm{V}}$$

$$\therefore \quad \log_{10}(D/\mathrm{pc}) \;=\; \frac{1}{5}\big(5 + \mathrm{V} - M_\mathrm{V} - A_\mathrm{V}\big) \;=\; 3.60$$

using $M_\mathrm{V} = -4.1$ for this star (from the table). So the distance is $D = 10^{3.60}$ pc = 4000 pc. The distance of the star is 4.0 kpc.

[This Q parameter is useful for early-type stars because it can be used to determine the spectral type, and hence surface temperature. However, it becomes less sensitive to spectral type (and therefore temperature) for later type stars. It is therefore is not used much for late-type stars.]